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(U) This report is aimed at developing the technology necessary to conduct cost effective and efficient validations of the sequencing of instruction used in the training of military occupational specialties. The overall goal covered by this final report was to determine the congruence of psychometric and instructional validation techniques for hierarchically ordered domains. This was done through two investigations in the course of two years. A total of 317 subjects were tested in the first project year on two algebra skill domains constructed from the Precision Measuring Equipment Curriculum of the Air Force Advanced Instructional System.

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6 A STRUCTURAL APPROACH TO THE
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TRAINING SEQUENCES.

9 FINAL REPORT,

June 1981

10 John R. Bergan, Principal Investigator
Anthony A. Cancelli, Co-Principal Investigator
Clement A. Stone, Research Associate
Olga Towstopiat, Research Associate

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Narrative Table of Contents

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Final Report Summary

This report is aimed at developing the technology necessary to conduct cost effective and efficient validations of the sequencing of instruction used in the training of military occupational specialties. The overall goal covered by this final report was to determine the congruence of psychometric and instructional validation techniques for hierarchically ordered domains. This was done through two investigations in the course of two years. A total of 317 subjects were tested in the first project year on two algebra skill domains constructed from the Precision Measuring Equipment Curriculum of the Air Force Advanced Instructional System.

Latent structure techniques recently developed by Leo Goodman at the University of Chicago were used to validate the hypothesized ordering between domains. The first step in the analysis was to construct a set of models representing hypotheses about the tasks under examination. The models developed for use in the present analysis assumed three basic classes of individuals for tasks in an hypothesized domain. These classes included masters of the skill represented in the domain, non-masters, and individuals in transition between non-mastery and mastery. Non-masters were characterized as failing all items in the domain, and masters as passing all items. Transitional individuals were assumed to respond inconsistently in a manner congruent with the assumption that they were still in the process of acquiring the concept or rule underlying mastery of the tasks in the domain under examination. Models asserting that tasks were in the same domain were compared to models asserting that the tasks were hierarchically ordered.

A Texas Instrument 745 terminal purchased for the two-year project was used in testing the extent to which the hypothesized models accurately repre-

sented the observed performance of the subjects. The analysis revealed two hierarchically ordered domains.

The finding of hierarchically ordered domains and the discovery that tasks within a domain may vary in difficulty level raise questions about generalization during the course of learning to master domain tasks. These questions may have far-reaching implications for training. More specifically, it may be possible to use information about difficulty level within a domain to determine where to begin instruction for the domain, and how to advance from one domain to the next. This possibility has significant implications for training efficiency.

In the second year of research, 626 subjects from the University of Arizona participated in an investigation designed to examine congruence of a math hierarchy validated psychometrically (Bergan, 1980; Resnick, 1973; Wang, 1973; White, 1973) and validated instructionally (White, 1974). The hierarchy consisted of quadratic and cubic equation tasks. Three studies comprised the investigation. Study 1 validated a hierarchy of quadratic and cubic equation tasks psychometrically. Studies 2 and 3 instructionally validated the domains and prerequisite relations identified in Study 1 and examined generalization and transfer between subordinate and superordinate skills.

The investigation incorporated latent structure techniques to validate the hypothesized domains and ordering between domains. New models were developed to represent response consistency in the contingency tables and to represent different types of transitional states of learning. The models for the second year were also able to distinguish between the prerequisite ordering of domains and ordering by difficulty within a domain.

The results of the three studies indicated a strong congruence between hierarchies validated psychometrically and instructionally. This finding suggests the possibility of employing the psychometric validation approach as a tool for gaining information about knowledge structures in a rapid manner. This information could then be used to effectively and efficiently develop tests for trainees.

It was also discovered that positive transfer occurred between the subordinate and superordinate skills in hierarchically ordered domains. The findings that generalization and transfer were evident between and within domains has important implications for training. They suggest that training could be targeted ahead of the learner's current level.

The statistical models employed in the second year of research could be used to develop tests that could not only sequence training content, but also could predict generalization and transfer within training sequences. These developments could greatly serve to increase training efficiency and reduce training costs.

Rationale for and Objectives of the Proposed Research

Since the time that Robert Gagné (1962) introduced his learning-hierarchy model in the early 1960's, there has been a growing recognition of the usefulness of empirically validated hierarchical learning sequences in teacher based, computer assisted, and computer managed training programs aimed at promoting the acquisition of basic math and science skills or at the development of performance capabilities related to various technical specialties pursued in military and industrial settings (Glaser, 1976; Glaser & Nitko, 1971; Glaser & Resnick, 1972; Nitko & Hsu, 1974; Resnick, Wang & Kaplan, 1973; White, 1973, 1974). However, despite the recognized usefulness of hierarchies, validated hierarchical sequences that can be applied in training are lacking. Moreover, there is at present no adequate, practical technology for conducting hierarchy validations. Unless such a technology is developed, the contribution that validated sequences could make to training will not be realized.

The validation of a learning hierarchy requires the testing of three hypotheses. One may be called the domain hypothesis and states that individuals respond in the same way to all items in a given domain of items. More specifically, the hypothesis holds that masters of the domain will tend to perform all items in the domain correctly while non-masters will tend to perform all items incorrectly. Some versions of the hypothesis make provisions for individuals in transition between non-mastery and mastery. Transitional individuals are assumed to display inconsistent performance on items in the domain. The second is that subordinate skills in a hierarchy are ordered prerequisitely with respect to superordinate skills (Gagné, 1977), and the third is that prerequisite skills mdeiate transfer for superordinate skills (Gagné, 1977).

The present project is designed to investigate research questions related to the testing of these hypotheses for the purpose of establishing guidelines that can be used in the development of a technology for hierarchy validation.

The Need for Validated Hierarchies

The need for validated hierarchies stems from their recognized potential value in training and from the fact that there are no adequately validated hierarchies in use in training programs today. Validated hierarchies could make two kinds of contributions in training. One of these relates to issues in instructional design, the other to assessment.

The Potential Role of Hierarchies in Instructional Design

The central advantage claimed for hierarchies in the area of instructional design has to do with the development of instructional sequences to facilitate transfer of learning. In numerous places in the literature, Gagné has advanced the view that lower level subordinate skills which are prerequisite to superordinate skills at higher levels in a hierarchy mediate transfer for the superordinate skills to which they are related (e.g., Gagné, 1962, 1968, 1973, 1977). The implication for instructional design is that instructional sequences should be arranged so that prerequisite skills are available to the trainee at the time that superordinate skills are to be mastered (Gagné, 1973).

Advocates of the learning-hierarchy view have pointed out that instructional sequences which ensure that prerequisite skills are available at the time of learning may produce highly beneficial results (e.g., Gagne, 1973; Glaser & Resnick, 1972). A sequence which takes into account prerequisite skills maximizes the likelihood that trainees will have appropriate prerequisite competencies at the time they are needed for superordinate-skill learning. On the other hand, a sequence developed without consideration for prerequisite relations leaves the question of whether or not trainees possess needed prerequisite competencies to chance. The result may be that some trainees will fail to master superordinate skills because they lack the prerequisites to superordinate skill mastery.

The Potential Role of Hierarchies in Assessment

The main advantage of empirically validated hierarchies with respect to assessment relates to the problem of adapting instruction to the needs of individual trainees. Given validated hierarchies, tests may be developed to individualize the placement of trainees in an instructional sequence (Glaser & Nitko, 1971; Nitko & Hsu, 1974; Resnick, Wang, & Kaplan, 1973). Placement tests based on validated hierarchies may be used in the initial phases of instruction to determine the point in an instructional sequence which will enable a trainee to encounter readily attainable goals and at the same time to avoid activities related to objectives that have already been mastered. In addition, placement tests may be used at the end of a sequence to determine what has been learned and thereby to establish what should be taught next (Nitko & Hsu, 1974).

The Current Lack of Validated Hierarchies

White and Gagné (1974) have noted that although the learning-hierarchy model has had some influence on the development of instructional materials it has not yet had the wide application that might have been expected. One apparent reason for the failure of the learning-hierarchy model to have a greater impact on training than it has had is that there are currently no adequately validated hierarchies that could be used in training programs.

During the period since Gagné (1962) introduced the learning-hierarchy model, there have been several studies attempting to validate isolated hierarchical sequences (White & Gagné, 1974). However, early investigations on hierarchies were marred by serious methodological flaws (White, 1973).

White (1973, 1974) suggested modifications in hierarchy validation procedures which eventuated in marked improvements in validation techniques. Despite these advances, adequate hierarchy validation has not yet been achieved.

As indicated in the initial paragraphs of this report, adequate hierarchy validation requires the examination of three hypotheses. Two of these hypotheses have never been effectively tested in hierarchical research.

The domain hypothesis has never been adequately tested in hierarchy investigations. A few attempts have been made to assess the assumption that prerequisite skills mediate transfer for superordinate skills, but much of the research in this area has had methodological flaws. Cotton, Gallagher, and Marshall (1977) reviewed the literature on the transfer hypothesis and have concluded that Gagné's transfer assumption has never been tested. Gagné's third hypothesis, the prerequisite-skills assumption, has been subject to intensive study (White, 1973). However, the validation procedures used to examine the prerequisite-skills assumption are methodologically flawed and are extremely time consuming and may not be suitable for broad scale application.

Advances in Statistics that Make a Practical Technology for Hierarchy Validation Possible

A major reason for the lack of progress in hierarchy validation described above is that until recently appropriate statistical procedures have not been available to test hypotheses germain to the development of effective, practical procedures for validating hierarchies. A number of procedures have recently become available which should make it possible to conduct hierarchy validations in a practical and effective way.

New Techniques for Validating Prerequisite Relations. During recent years Gagné's prerequisite-skills assumption has served as a focal point for efforts to develop statistical procedures for use in hierarchy validation. White (1973) has shown that techniques used to assess prerequisite relations by Gagne and his colleagues in early hierarchy research were inadequate in that they failed to provide a statistical test for prerequisite associations which took into account errors in measurement. More recent research on prerequisite relations using a variety of scaling techniques including scalogram analysis (Guttman, 1944), multiple scalogram analysis (Lingoes, 1963), and the ordering theoretic method (Bart & Airasian, 1974; Bart & Krus, 1973) has been faulted on similar grounds. None of these procedures provides a suitable statistical test for prerequisite relations (Airasian, Madaus, & Woods, 1975; Dayton & Macready, 1976; White, 1974).

During recent years a number of attempts have been made to develop procedures to test Gagné's prerequisite-skills hypothesis statistically (Emrick & Adams, Note 2; Murray, Note 3; Proctor, 1970; White & Clark, 1973). Dayton and Macready (1976) have shown that each of these procedures represents a special case of a general latent-structure model which has the advantage of being capable of testing for prerequisite relations in both linear and nonlinear hierarchies. Goodman (1974) has also developed a latent-structure approach which can be used to test for prerequisite orderings in linear and nonlinear hierarchies.

New Techniques for Validating Positive Transfer. Although attempts to establish statistical techniques for use in hierarchy validation have focused mainly on Gagné's prerequisite-skills hypotheses, the need for procedures to examine Gagné's second major hypothesis, the positive transfer assumptions, are equally great. A recent review by Cotton, Gallagher, and Marshall (1977) attests to this fact. As indicated above, these investigators failed to find a single published study which provided a suitable test of Gagne's positive transfer assumption. Bergan (1980, in press) has shown that structural equation models based on Sewall Wright's (1921, 1960) pioneering work in path analysis can be used to assess positive transfer in a learning hierarchy.

Structural equation procedures based on regression analysis (Joreskog & Sorbom, 1978) are available for use with interval scale dependent measures (Duncan, 1975; Heise, 1975). In addition, Goodman (1972, 1973a, 1973b) has developed structural equation techniques involving the use of log linear models (Bishop, Fienberg, & Holland, 1975) that can be applied with dichotomous and polytomous scores of the types typically used in hierarchy validation.

New Techniques for Domain Validation. Gagné (1977) assumes that the skills in a learning hierarchy represent response classes rather than discrete behavioral capabilities. For example, within the learning hierarchy viewpoint, it is assumed that a trainee who possesses a skill such as multiplying two mixed numbers will be able to use that skill to solve a broad range of similar problems.

One of the major problems in hierarchy validation is to determine whether or not the items on a test of skill performance measure the trainee's ability to perform the full range of behaviors included in the response class assumed to be represented in the skill under examination. Hively (1974) uses the term item domain to refer to the class of items associated with a given skill. Hively, Patterson and Page (1968) developed a set of rules for generating test items falling within various domains. Since the early work of Hively and his associates, other investigators have elaborated on the concept of item domain and have attempted to develop item generating procedures for various types of domains (Shoemaker, 1975).

Although awareness of the need to determine empirically the extent to which specific test items represent an item domain has existed for some time, statistical procedures for empirically validating item domains associated with different skills have been lacking. For example, White (1974) in an article on hierarchy validation, discussed the need for determining statistically the extent to which different items assessed the same skill, but was forced to conclude that there were no available statistical procedures for making such a determination.

The Goodman (1974) and Dayton and Macready (1976) latent structure procedures are suitable for use in empirically validating an item domain. For instance, to test the hypothesis that a set of items belong within the same domain using the Goodman latent structure technique, one could hypothesize a model composed of three latent classes. One of these would represent those learners who had acquired the skill being assessed by the items in the domain under investigation. Trainees in this group would be expected to pass all domain items presented to them. The second type would represent learners who had not acquired the skill in question. Trainees in this group would be expected to fail all domain items which they encountered. The third class would be composed of individuals in transition between non-mastery and mastery. Either the chi-square goodness-of-fit or likelihood-ratio statistic can be used to test the fit of a model of this type to a set of data collected on item performance in the domain targeted for study.

A Structural Approach to Hierarchy Validation. The present research combines use of the Goodman (1974) latent structure techniques with structural equation procedures (Goodman, 1973a) in what may be termed a structural approach to hierarchy validation. The research examines the validity of item domains in a hierarchy and addresses both Gagné's prerequisite ordering and positive transfer hypotheses as these assumptions relate to the task of developing practical procedures that can be applied in hierarchy validation in domain-referenced assessment and training design. The hierarchical relations selected for examination involve basic algebra skills included in military training. The specific skills targeted for study were designed to be congruent with the Precision Measuring Equipment Curriculum of the Advanced

Instructional system (AIS), an individualized training program operated by the Airforce at Lowrey Airforce Base. Analysis of these skills in the present project not only affords general guidelines for the validation of military training sequences, but also provides direct information that could be used to improve the efficiency and effectiveness of military training involving basic algebra skills.

Hierarchy Research Needs

Although adequate statistical procedures for examining hierarchical relations are now available, information is lacking on how to go about the validation process. Three kinds of research needs must be met before it will be possible to determine the most efficacious procedures for validating hierarchical associations. One of these involves the issue of how skills should be measured in validating the prerequisite-ordering hypothesis. The second has to do with skill measurement in validating the positive-transfer hypothesis, and the third deals with domain validation in hierarchical sequences.

Needs Related to Prerequisite-Skills Validation. One of the initial steps in hierarchy validation is to test for hypothesized prerequisite relations in the hierarchy under examination. Two strategies have been suggested for accomplishing this task. Research is needed to determine whether or not these two procedures yield different results.

One of the strategies used in prerequisite-skills validation is the psychometric approach (Resnick, 1973; Wang, 1973). In this approach, trainees are tested on skills under examination in a hierarchy, and a statistical procedure is applied to determine the existance of prerequisite dependencies. Some years ago White (1973) criticized the psychometric approach on the grounds that it does not control for random forgetting. White took the position that skills in a hierarchy may be forgotten in a different order than the order in which they are learned.. In accordance with this position, White (1974) argues that validation of the prerequisite-skills hypothesis requires a validation procedure in which learners who do not initially possess the skills in a hierarchy are taught the skills. He further suggested that testing for skill acquisition should be conducted during the course of learning rather than when instruction has been completed.

In support of the assumption of random forgetting, White cited only one study, an early investigation by Gagné and Bassler (1963). There are a number of reasons why the Gagné and Bassler study does not provide convincing evidence for the random forgetting assumption. First, adequate statistical procedures for testing the prerequisite skills hypothesis were unavailable at the time of the Gagné and Bassler investigation. Thus, it is not certain that all of the prerequisite relations that were assumed to be shown by the data actually did exist (White, 1976). Second, at the time of the investigation, there were no statistical techniques to assess the extent to which observed differences between learning and retention reflected measurement error as opposed to forgetting. Finally, the retention test which Gagné and Bassler used involved items which were different from the tiems used to assess learning. Thus, what Gagne and Bassler called a retention test could also be described as a test of generalization.

Recognition of the lack of convincing evidence provided by the Gagné and Bassler study has recently led White (1976) to suggest that the psychometric procedure ought to be reconsidered for use in hierarchy validation. The widespread application of hierarchical sequences in military training will require the validation of vast numbers of hierarchies. The psychometric approach to testing the prerequisite-skills hypothesis is much more efficient than the instructional strategy advocated by White. If it were possible to use the psychometric approach in the validation process and attain accurate results, a huge savings in time and personnel would be realized . In view of the superior efficiency of the psychometric approach and the lack of convincing evidence contra-indicating the use of the approach, research to assess the efficacy of the psychometric technique is clearly warranted. In this regard, there is a need to determine the extent to which hierarchical models validated under White's instructional strategy match models validated psychometrically. The present project is designed to meet this research need.

As indicated in the discussion of the Gagné and Bassler study, the extent to which skills are retained in the order in which they are learned has implications with respect to the utility of the psychometric approach. In order to establish fully the utility of the psychometric validation strategy there is a need for research on the question of whether or not skills are forgotten in a different order than the order in which they are learned. The present project addresses this research need.

Needs Related to Positive Transfer Validation. As indicated above published studies assessing Gagné's positive transfer hypothesis are lacking. One possible reason for this lack is that procedures advocated for testing positive transfer are difficult and time consuming to implement. Many investigators, particularly those studying complex hierarchies involving many connections have dealt with the issue of transfer by ignoring it and focusing instead on the validation of prerequisite relations (White & Gagné, 1974).

Validation of Gagné's positive transfer hypothesis has generally been conceptualized within a transfer-of-training paradigm. White & Gagné (1974) suggest a validation strategy which illustrates this fact. The White & Gagné approach involves the following steps: First, choose as many prerequisite relations in the hierarchy under consideration as can be examined within existing constraints on time and resources. Second, for each connection to be studied, identify groups of learners who possess all relevant prerequisite skills, but who lack the specific prerequisite and superordinate skills targeted for study. Third, conduct a standard transfer-of-training experiment in which half of the learners receive training on the superordinate skill. Positive transfer is indicated if learners receiving prerequisite skill training perform significantly better on the superordinate skill training task than learners who do not receive prerequisite skill instruction.

As indicated above, Bergan (1980, in press) has shown that Gagné's positive transfer hypothesis can be tested using structural equation models. Within a structural equation approach, direct and indirect effects among a set of variables can be examined in the absence of an experiment involving random assignment of individuals to treatment conditions (Duncan, 1975; Goodman, 1972; Heise, 1975). For example, in the case of interval scale data, the direct effects of one variable on another can be assessed using ordinary least squares regression techniques (Duncan, 1975). The magnitude of the direct effect of the first variable on the second is given by a structural coefficient which in ordinary least squares regression analysis is the regression coefficient in the regression equation.

A structural approach to testing Gagné's positive transfer hypothesis is potentially more efficient than the procedure suggested by White and Gagné (1974). The increased efficiency derives from the fact that structural equations can be used with the same data collection procedures as those employed in prerequisite skills validation. Thus, for example, structural equations can be used to examine positive transfer using White's (1974) instructional procedure for prerequisite skills validation. White's instructional procedure requires less time and is more practical to implement than the White & Gagné (1974) transfer paradigm in that it necessitates only one group of learners who are taught all skills in a linear sequence whereas many groups learning different skills are needed to implement the White and Gagné transfer procedure.

Structural equations can be used to achieve an even greater gain in efficiency than that associated with the use of the White instructional technique if they are coupled in positive transfer validation with the psychometric validation procedure. The psychometric procedure is, of course, much more efficient than the White and Gagné approach in that all that is required to implement the technique is to test a group of trainees.

To apply structural equations to test the assumption that prerequisite skills mediate transfer for superordinate skills, prerequisite and superordinate skills must first be identified. This can be accomplished using prerequisite skills validation procedures discussed above. After prerequisite and superordinate skills have been determined, a structural model comprised of equations expressing hypothesized effects of previously validated prerequisite skills on superordinate skills can be constructed. Data from either the White instructional procedure of the psychometric procedure can then be used in testing model-data fit.

It is possible that structural equations used with the psychometric procedure would not yield the same results as would be attained using White's instructional paradigm. If this were to occur, it could be argued that White's paradigm provided a more valid demonstration of transfer than a structural equation approach using psychometric validation procedures in that the White paradigm involves learning, whereas the psychometric approach does not. However, if psychometric procedures could be assumed to yield the same transfer relations as identified through the White paradigm, then a substantial gain in efficiency could be attained in the validation process.

Research is needed to determine the extent to which structural equation techniques coupled with instructional or psychometric validation procedures reveal the same transfer relations. The present project is designed to meet this research need.

Needs Relating to Domain Validation. The validation of item domains is an essential precursor to adequate examination of the other major hypothesis involved in hierarchy validation. Without domain validation, it is impossible to determine the extent to which test items reflect the response classes that they are assumed to represent. In the absence of domain validation, failure to confirm either prerequisite skills or positive transfer hypotheses could be attributed to the possibility that the specific items used in validation did not adequately represent hypothesized classes for the skills under investigation.

The empirical determination of relations among tasks within domains requires the construction of models to represent item domains. A number of models assume some kind of equivalence relation among tasks in an item domain. That is, they all assume that tasks will tend to be responded to in the same way by at least some groups of individuals. For example, Dayton and Macready (1976) have conceptualized item domains in terms of models that assume a mastery class composed of individuals who tend to perform all domain tasks correctly and a non-mastery class comprised of individuals who tend to fail all tasks in the domain. By contrast, Bergan, Cancelli and Luiten (1980) have described models based on Goodman's (1975) work in response scaling that assume three classes of individuals in a homogeneous domain, non-masters, masters, and what Goodman (1975) calls unscalable individuals. Masters are assumed to perform all tasks in the domain correctly while non-masters are assumed to fail all tasks. Individuals in the unscalable category tend to manifest responses inconsistent with non-mastery or mastery and may be thought of as being in a transition state between non-mastery and mastery.

Varying assumptions may be made about task difficulty (i.e., the probability of accurate performance) within mastery, non-mastery, or transitional classes. More specifically, it may be assumed that task difficulty varies within classes or that it is equal across tasks within classes. For example, consider two algebra tasks shown empirically to belong in a domain characterized by problems in which a common term, say x, has to be factored from an expression such as (xa + xb). Suppose that the tasks were similar in all significant respects except that one necessitated three steps to achieve a solution and the other required only two steps. Suppose further that a model including masters, non-masters, and transitional individuals were used to describe relations among the tasks in this domain. Under this kind of model, masters would be assumed to perform all tasks correctly. For masters, the two tasks would be equally difficult in that the same proportion of individuals (i.e., all individuals) would display mastery of each task. Since non-masters would be assumed to fail all tasks, the tasks would also be equally difficult for them. By contrast, the tasks could vary in difficulty for transitional individuals. It would be reasonable to assume that the problem requiring three steps for solution would be more difficult than the problem requiring two steps for transitional individuals.

The possibility of within domain variations in task difficulty suggests that in a certain sense there may be sequential ordering within domains as well as between hierarchically related domains. As already indicated, the tasks within a domain are assumed to be equivalent, but equivalence may not always imply complete symmetry. Tasks that vary in difficulty for a given class such as that of transitional individuals may be thought of as being asymmetrically equivalent. Sets of asymmetrically equivalent tasks may be ordered by difficulty to form a sequence within a domain. Nothing i known about the conditions that may produce asymmetrical equivalence relations within a domain. The present report examines the hypothesis that tasks within domains comprised of algebra problems will form asymmetrical equivalence relations congruent with variations in the number of steps required to achieve problem solution.

The presence of an ordered relation between tasks provides one criterion that can be used to establish boundaries between domains. The concept of domain boundaries is, of course, essential in delimiting the content of a domain. Nonetheless, it is not necessary to think of boundaries as impermeable walls. Domains may include large numbers of tasks, and it is quite possible that some inter-domain task comparisons may suggest boundary permeability. For example, suppose that a group of item sets were used to assess performance on three academic tasks, _A_, _B_, and _C_. Assume that task _A_ was shown to be asymmetrically equivalent to task _B_ and that task _B_ was found to be asymmetrically equivalent to task _C_. In addition, suppose that an ordered relation were observed in which _A_ was found to be subordinate to _C_. In a case such as this, _A_ and _C_ would be in separate domains, but _B_ would be in both the _A_ domain and the _C_ domain. Thus, the boundary between the _A_ domain and the _C_ domain would be permeable. The present report examines the possibility of permeability in domain boundaries. In this connection it is hypothesized that if permeability does exist, it will occur between tasks at the higher levels of a subordinate domain and the lower levels of the related superordinate domain.

First Year Research

This section of the report details studies conducted during the first project year. These involved the psychometric validation of algebra problem solving domains (Task 1) and the psychometric validation of the hierarchical ordering of domains (Task 2).

Objectives For The First Project Year

Objectives for the first year of the project focused on the attainment of Task 1 objectives. These include both outcome and enabling objectives.

Outcome Objective. To validate psychometrically the domains and the ordering of item domains for algebra tests selected from an examination of the Precision Measuring Equipment Curriculum.

Enabling Objectives.

a. to construct and write item domains for each hypothesized domain.

b. To task analyze algebra skills from psychometrically validated domains selected from the Precision Measuring Equipment Curriculum.

c. To construct a domain referenced test of items randomly selected from each domain.

d. To administer the test to approximately 200 subjects.

e. To score responses.

f. To construct and test latent class models to determine the extent to which hypothesized models fit (i.e., accurately represent) observed test performance.

Method

Subjects. The subjects were 317 volunteers from a high school and university in the Southwest selected to represent a wide range of skill levels in solving algebra problems. Subjects ranged from high school freshmen taking a first course in basic mathematics to university students, a number of whom had had college math courses. There were approximately equal numbers of males and females representing a broad spectrum of ethnic backgrounds. Approximately 88% were Anglo, 8% were Mexican-American, and 4% were divided among Blacks, native American Indians, and Asians. More subjects were used than the 200 originally intended for the study so that the full range of algebra skills likely to be present in military trainees would be represented.

Tasks. A group of algebra tasks hypothesized to form an ordered set of behavioral domains was selected for use in conducting domain structure analysis. Algebra was chosen because it is a highly structured content area. The structured nature of the discipline facilitated the formulation of hypothesized domains and domain orderings.

An adaptation of facet analysis (Berk, 1978; Millman, 1974) was used in formulating hypothesized domains and domain orderings. Facets were defined as classes of behavioral operations involved in performing algebra tasks. Three facets were identified for this study: transposition of terms, application of the distributive property, and factoring. Each facet was hypothesized to represent a homogeneous item domain.

Problems within each domain varied in terms of the number of steps required to achieve problem solution. For example, some problems could be solved in a single step such as multiplying both sides of an equation by one term or expression. Other problems required as many as five steps for solution. It was assumed that item sets within each domain would form asymmetrical equivalence relations sequenced in accordance with the number of steps necessary for problem solution.

The hypothesized domains identified in the study do not represent independent dimensions. For example, it is impossible to solve factoring problems without transposing terms. The inclusion of operations defining one domain in problems reflecting another domain suggested an ordering of the domains congruent with Gagné's (1962, 1977) view that component tasks form an ordered sequence. An examination of the hypothesized domains to identify components suggested that the term-transposition domain would be subordinate to both the distributive property and factoring domains.

Problems illustrating the hypothesized domains are shown in Table 1 in Appendix A.

The first domain included problems requiring the transposition of terms from one side of an algebra equation to the other. Transposition was effected by one or more arithmetic operations (e.g., multiplication or subtraction). For instance, the first problem shown in Table 1 for this domain required transpositing the term A to the right side of the equation by multiplying both sides of the equation by A. The second domain involved applcations of the distributive property in which a single term had to be multiplied with each of two terms in an expression. The third domain required factoring a common term from an expression. For example, in the problems in Table 1, X must be factored from expressions including the terms N and R. Factoring is regarded in algebra texts as an application of the distributive property. This application involves a reversal of the multiplication operations carried out in using the distributive property.

Each of the three hypothesized domains involved problems representing an ordered set of elements. Ordering was based on the number of steps required for problem solution. For example, the first problem shown in Table 1 for the term transposition domain required only one step to achieve problem solution. By contrast, the second problem required two steps.

Variations in number of required steps were by necessity different for different domains. For example, the simplest factoring problem required two steps for solution. First a common term X had to be factored from an expression. Then the expression had to be moved to the right side of the equation. The term transposition domain contained two step categories: one-step problems and two-step problems. The distributive property domain contained three step classes: three-step problems, four-step problems, and five-step problems. The factoring domain contained the largest number of step categories. Factoring problems ranged from two steps to five steps.

Test Construction and Scoring. Following the facet analysis, item forms and item form shells (Hively, Maxwell, Rabell, Sension & Lundin, 1973) representing each of the domains and step categories within domains were constructed. The item forms provided descriptions of the classes of problems to be solved, stimulus and response characteristics of those classes, and cell matrices indicating class variations. The item form shells indicated materials, directions, scoring specifications, and replacement rules for generating items. The item form approach was used because it makes it possible to represent the population of problems in a domain in a precise fashion.

Test items were constructed to correspond to item form specifications. Two items representing identical problems were prepared for each type of algebra task included in the study. These items varied only in the specific letters used to represent equation terms. This made it possible to reflect variations in response consistency in the models used to assess domain structure.

Each pair of terms representing a task was scored *1*, *2*, or *3*. A *1* indicated that neither of the two items was answered correctly. A *2* indicated that one of the two item pairs was answered correctly, and a *3* indicated that both items were responded to correctly.

Procedures. Testing was carried out in groups of about thirty. The participants were told that the purpose of the study was to determine how people solved algebra problems. After the test booklets were passed out, the experimenter gave instructions for responding to the test. Trainees were instructed to solve the algebra problems presented and to write their solutions in the test booklets provided. Trainees were instructed to attempt all problems and to provide solutions even in cases in which they were unsure of the answers. Following the instructions the trainees were told to begin the test and were assured that they would have as much time as necessary to complete the problems. During the course of the testing, the experimenter and an assistant monitored each subject's performance to insure that the task was understood. The vast majority of the subjects comprehended what they were to do on the basis of the initial instruction. However, in one or two cases there were some questions. When this happened, the experimenter simply repeated the instructions for the individual having difficulty. In all cases the repeated instruction was sufficient to enable the individual to respond to the questions.

The Latent Class Approach

Latent class models (Goodman, 1974) were used to assess equivalence and ordered relations among the algebra tasks examined in the study. Latent class models explain association in a contingency table in terms of a latent (i.e., unobserved) variable or set of latent variables each of which includes a set of latent classes. For example, in the present research latent class models were constructed to reflect variations in task mastery. The latent variable in this case was mastery variations. This variable included different latent classes, such as a mastery class and a non-mastery class. A latent class model can be used to generate maximum likelihood estimates of expected cell frequencies which indicate expected response patterns under the assumption that the model being examined is true. A brief description of latent class techniques is provided in Appendix C.

Latent class models are tested by assessing the correspondence between observed cell frequencies and estimates of expected cell frequencies using the chi-squared statistic. When the correspondence between observed and expected frequencies is close, the value of X^2 will be low and the model being tested can be said to provide an adequate fit for the data. Clifford Clogg (Note 3) has developed a computer program that carries out the iterative process used to generate maximum likelihood estimates of expected cell frequencies and that computes the X^2 value to test the fit of a model to a data set. Clogg's program was used in the present investigation.

Models Tested

The latent class models initially designed for the present project were intended to distinguish between ordered and equivalence relations among algebra tasks. For reasons to be discussed, these models were significantly modified for the second year research. To understand why the models were designed as they were, it is necessary to understand model distinctions involving the ordering and equivalence of tasks. Consider Table 2 in Appendix A, cross-classifying performance on two items. Thus, a subject's score for each task may fall into one of three categories, zero right, one right, or two right. These categories can be designated by the numbers 1, 2, and 3 respectively.

In a table of this kind, a score of 1 on each task would suggest non-mastery. This response pattern would be reflected in the 11 cell in the table. A score of 3 on each task would suggest mastery. This pattern is reflected in the 33 cell. A score of 3 on task A and 1 on task B would indicate mastery of task A without evidence of mastery of task B. Scores of 2 would reveal inconsistent performance characteristic of transition between non-mastery and mastery. Since the items for each task are identical, scores of 2 should reflect errors which ought to occur at a relatively low frequency.

Given an ordered relation between tasks A and B, the number of responses in the 31 cell should be significantly greater than the number in the 32 cell. Under the assumption of ordering, a build-up would be expected in the 31 cell indicating that a significant number of subjects had mastered A without having begun to master B. The 32 cell would be expected to have relatively few responses because the 2 category represents response inconsistency for task B.

If the tasks were equivalent, the number of individuals in both the 31 and 32 cells would be small since both these cells would reflect response inconsistency. The relation between the 31 and 32 cells would not be crucial so long as the probability for the 31 cell was not larger than the probability for the 32 cell. Two relations between the 31 and 32 cells could occur without contraindicating the equivalence assumption. Either the cells could be equiprobable or there might be a significantly greater number of individuals in the 32 cell than in the 31 cell.

As this discussion shows, a critical issue in determining whether two tasks form an ordered or equivalence relation is that of determining whether the hypothesis that the occurrence of responses in the 31 and 32 cells is equiprobable is supported by the data. If this hypothesis is rejected, it is necessary to determine whether the probability of a response in the 31 cell is greater than the probability of a response in the 32 cell for masters of task A. If this turns out to be the case, a model describing an ordered relation between the tasks may be considered. If the probability for the 31 cell is not greater than the probability for the 32 cell, an equivalence model may be suggested to represent the data.

Eight latent class models were examined in the study. The models are described in the following paragraphs and are displayed visually in Table 3 (App.A). The E's and curved lines in the visual display indicate cells constrained to be equiprobable under a given model. The I's indicate cells for which the assumption is made that the probability of a given response level on task A is independent of the probability of any particular response level on task B. The X's indicate response patterns associated with specific latent classes. For example, the X in the 11 cell of H_1 indicates the association of the 11 response pattern with the non-mastery latent class.

The Independence-Equiprobability Model. The first model, designated H_0, asserts independence between task pairs and equiprobability between categories 1 and 2 for the task assumed to be the least difficult in the task pair. This model served as a standard against which to compare the other models tested. The equiprobability provision was included to make the model congruent with models being examined. As mentioned earlier, the central criterion for distinguishing between ordered and equivalence relations is one asserting equiprobability between certain task categories. The equiprobability provision was included in model H_0, as well as some of the other models examined, to provide a basis for distinguishing between ordered and equivalence relations. If there had been any instances in which model H_0 provided an adequate description of tasks in the domain under examination, the hypothesis that the tasks were not related would have been supported.

The Model of Symmetry. Model H_1 asserted symmetrical equivalence between tasks. Model H_1 included 6 latent classes: a non-mastery class, a partial mastery class, a mastery class, and 3 transition classes reflecting symmetrical inaccuracies in responding. The 3 classes assuming inaccurate responding each asserted equiprobability for one pair of cells in the table cross-classifying the tasks under examination. For example, one of these classes asserted that the probability of the 12 cell would be equal to the probability of the 21 cell. The second asserted that the probability of the 13 cell would be equal to the probability of the 31 cell, and the third assumed that the probability of the 23 cell would be equal to the probability of the 32 cell. Because of the symmetrical nature of its equiprobability restrictions, this model has been described in the literature as the model of symmetry (Bishop, Fienberg, & Holland, 1975). The model of symmetry implies equal item difficulty for the tasks under examination. Tasks for which this model provided an adequate fit for the data were described as being symmetrically equivalent.

Asymmetrical Equivalence Models. Model H_2 included 3 latent classes, a mastery class, a non-mastery class, and an unscalable class composed of transitional individuals. Model H_2 assumed that masters would respond correctly to all problems presented to them. Thus, in the mastery class the probability of the 33 response pattern was restricted to be 1. Similarly, the model assumed that non-masters would fail all problems. Thus, in the non-mastery class the probability of the 11 category was restricted to be 1. It was presumed that in the unscalable category, the probability of a particular level of performance on one task would be independent of a given level of performance on the other tasks, and that the 1 and 2 categories would be equiprobable for one of the tasks. The equiprobability restriction was included as a criterion for distinguishing between equivalence and ordered relations for reasons already discussed.

Model H_2' is a special case of model H_2. It is like model H_2 in all respects except that is does not include the equiprobability restriction imposed under H_2. Model H_2' was included to reflect the fact that two tasks may be equivalent even though the 1 and 2 categories of the more difficult task are not equiprobable. It may happen that the probability of a response in the 32 cell is greater than the probability of a response in the 31 cell. This is exactly the opposite of what is to be expected under the hypothesis of an ordered relation between tasks. When the hypothesis of equiprobability is rejected, but the probability of the 32 cell is greater than the probability of the 31 cell, it is appropriate to test models which assert equivalence, but which do not include equiprobability restrictions. Model H_2' is one such model.

Model H_3 included 4 latent classes, a non-mastery class, a partial mastery class, a mastery class, and an unscalable class. The partial mastery class was similar to the unscalable class in that both reflected less than completely accurate responding on the part of examinees. However, model H_3 asserted that individuals in the partial mastery class consistently performed 1 out of 2 problems correctly on both tasks under examination for a given task pair. More specifically, the partial mastery class asserted that for members of that class the probability of getting 1 out of 2 items correct for both tasks would be 1. The unscalable class did not assume this kind of consistency in partially accurate responding.

Model H_3' assumed four latent classes, a non-mastery class, a partial mastery class, a mastery class, and an unscalable class. The restrictions for non-mastery, partial mastery, and mastery classes were the same as those given for H_3. Moreover, similar restrictions were imposed for partial mastery.

Model H_3' differed from H_3 because it did not impose an equiprobability restriction in the unscalable category. The concept of partial mastery implies a significant number of individuals who get 1 problem right. Given this state of affairs, not only should a build-up of individuals in the 22 category be expected, but also it would not be unreasonable for the probability of occurrence of the 32 category to be greater than the probability for the 31 category. Model H_3' reflects the fact that equiprobability need not always occur in a model asserting equivalence between tasks.

Model H_4 is very similar to H_2. The difference between the two is related to the equiprobability restriction in the unscalable class. In asserting both independence and equiprobability, model H_4 necessarily makes the 21 and 22 cells as well as the 31 and 32 cells equiprobable in the unscalable latent class. Equiprobability does not obtain for the 11 and 12 cells because the 11 cell represents a separate latent class, i.e., the non-mastery class. Model H_4 restricts equiprobability in the unscalable class to the 31 and 32 cells. This is accomplished by making the 21 cell represent a separate latent class. The probability of the 21 response pattern in this class is restricted to be 1. The effect of this is to make the observed and expected cell frequencies for the 21 pattern equal. Thus the pattern contributes nothing to the value of X_2. With the exception of the restriction on the 21 cell, model H_4 is exactly the same as H_2. Like H_4, it contains mastery, non-mastery and unscalable latent classes. Moreover, the restrictions on the mastery and non-mastery classes are the same as those for H_2. The unscalable category assumes independence between tasks with the 21 pattern ruled out of consideration. In addition, it asserts equiprobability for the 31 and 32 cells.

An Ordered Relation Model. Model H_5 asserted an ordered relation between task pairs. This model contained four latent classes, a non-mastery class, an unscalable class, a mastery class and a subordinate task mastery class. The restrictions in the non-mastery and mastery classes were identical to those used in the equivalence models. Independence was assumed in the unscalable class. In the subordinate task mastery class the probability of passing both subordinate task items was assumed to be 1. The probability of passing both superordinate task items was assumed to be zero and the probabilities of getting no correct responses and 1 correct response on the superordinate task were set equal to the observed proportions of responses in those two categories. The last of these restrictions was imposed so that all individuals who had mastered the subordinate task including those in transition toward superordinate task mastery would be included in the latent class reflecting subordinate task mastery.

Results

Within Domain Results. Results of the model testing within-domains revealed two domains instead of the three hypothesized. The factoring and distributive property problems turned out to be in one domain. Tables 4 and 5, in Appendix A, present the observed responses for the cross-classification of every possible task pair for each of the two domains. Table 4 shows the cross-classification for the term transposition domain while Table 5 displays the cross-classification for the Distributive Property-factoring domain. In Table 4 the letters indicate the addition-subtraction (A) and multiplication-division (M) dimensions. In Table 5 they stand for factoring (F) and distributive property (D) problems. Numbers in both tables represent the number of steps required for problem solution.

The response patterns in the tables indicate various combinations of the number of correct responses for each task pair examined. For example, the 11 pattern indicates no correct responses on either task while the 33 pattern represents 2 correct responses for each task. Note the large number of responses falling in the 11 and 33 categories in the tables. These patterns represent the critical cells for establishing equivalence relations. Notice further that most task sets have about the same number of individuals in the 31 and 32 cells. The 31 cell represents individuals who have mastered one task, but have not begun to acquire the second task. As already indicated, given an ordered relation between tasks, the number of individuals in the 31 cell would be expected to be larger than the number of individuals in the 32 cell. On the other hand, given an equivalence relation between the tasks, the number of individuals in both the 31 and 32 cells would be expected to be small.

Tables 6 and 7, in Appendix A, present the results of model testing for the hypothesized domains. In the model testing process, all possible pairs of tasks within a given domain were compared. Table 6 shows the chi-squared tests for all possible task pairs in the term transposition domain. The letters designating tasks refer to the addition-subtraction (A) and multiplication-division(M) dimensions for this domain. The numbers refer to the number of steps required for problem solution. For example, 1 refers to a problem requiring only one step for solution.

The model testing process required the selection of a preferred model based on statistical comparisons among various models examined. To illustrate the comparison process, consider the results for H_0 and H_2 for the A1-M1 task pair given in Table 6. The X^2 value for model H_0 is 200.65 with 5 degrees of freedom, which is significant well beyond the .01 level. The X^2 value for model H_2 is 1.18 with 3 degrees of freedom which has a p value of about .90. Model H_0 and H_2 are hierarchical. That is, H_2 contains all of the characteristics of H_0 plus 2 additional characteristics. These additional characteristics reflect the inclusion of a mastery and non-mastery latent class under H_2. Model H_2 has 3 degrees of freedom, whereas H_0 has 5. The loss of 2 degrees of freedom reflects the inclusion of the non-mastery and mastery latent classes. Because H_0 and H_2 are hierarchical, they can be compared statistically (Goodman, 1974). The X^2 for H_2 can be subtracted from the X^2 for H_0. The result will be an X^2 with 2 degrees of freedom. In the case of the A1-M1 task pair, the subtraction of H_2 from H_0 yields an X^2 of 198.47 with 2 degrees of freedom, which is significant far beyond the .01

level. Model H_2 provides an excellent fit for the data. Moreover, none of the models improve over H_2. Consequently, H_2 was selected as the preferred model for the A1-M1 task pair. Not all of the models in Table are hierarchically related. For example, H_1, the symmetry model, is not hierarchically related to either H_0 or H_2. Consequently, it is not possible to compare H_1 directly with H_0 or H_2.

The results on Table 6 show that in no case did model H_0 or H_1 provide an acceptable fit for the data. Consequently, the hypothesis that the task pairs under examination were unrelated and the hypothesis that they were symmetrically equivalent could be rejected for all of the tasks investigated.

In all cases except one, one of the asymmetrical equivalence models provided an acceptable fit for the data. In some instances, the model including an equiprobability restriction provided an adequate fit. In other cases, for example in the case of task pair A1-A2, the equiprobability assumption was rejected. However, the probability of being in the 32 cell was found to be higher than the probability of being in the 31 cell. Consequently, it could safely be concluded that the tasks for this pair were not ordered.

The one instance in which the hypothesis of equivalence relations was rejected was that involving the A1-M2 task pair. The two tasks involved in this comparison represented marked differences in difficulty level within the item domain. The one-step addition problem was the simplest task in the domain, whereas the two-step multiplication problem was among the most difficult. Model H_5 provided an acceptable fit for these two tasks indicating an ordered relation between them. The ordered relation for the A1-M2 task pair suggests permeability in domain boundaries. Tasks A1 and A2 are in the same domain. A2 and M2 are not in the same domain. The fact that A1 and M2 are found to be in separate domains suggests that the boundaries between domains may not be rigid.

The results for the term transposition domain reflect a highly consistent pattern. As already indicated, the hypothesis of asymmetrical equivalence was supported in every instance except one. The asymmetrical equivalence observed in the domain reveals a structured arrangement of tasks. The tasks requiring two steps for problem solution are more difficult than those requiring only a single step.

Table 7 shows the results of model testing for the combined distributive property-factoring domain. The letters in Table 7 refer to factoring problems (F) and distributive property problems (D), and the numbers indicate the number of steps required for problem solution.

As in the case of the transposition domain, the results for the combined distributive property-factoring domain reveal a highly consistent pattern. In most instances, one of the asymmetrical equivalence models provides a suitable fit for the data. However, in some cases, the model of symmetry fit the data to an acceptable degree. This suggests that at the higher levels of algebra skill, problems are more likely to be equivalent for all groups of individuals, including those in transition. This is understandable since those in transition with respect to higher level skills bring a broad background of subordinate skills to the task of solving higher level factoring and distributive property problems.

In only one case did a task not form an equivalence relation with other tasks. This was the case for the most difficult factoring task. Model H_5 provided an acceptable fit for comparisons involving this task. Model testing revealed that this task was superordinate to all of the other distributive property and factoring tasks. Analysis of the characteristics of the task revealed that it required not only factoring, but also application of the multiplication operations used in distributive property problems. This suggests the existence of a superordinate domain hierarchically related to the factoring-distributive property domain. Further research is needed to investigate this possibility.

Between Domain Results. Table 8, in Appendix A, presents observed response patterns for the cross-classification of tasks representing the term transposition and factoring-distributive peroperty domains. Note the large number of individuals attaining the 31 response pattern and the relatively small number of individuals for the 32 pattern. This is what is to be expected under the hypothesis of an ordered relation between task pairs.

Table 9 in Appendix A, shows the chi-squared tests for the cross-classifications in Table 8. In all cases model H_5 afforded an acceptable fit for the data, and in all cases except four model H_5 was preferred over the other models tested. Two equivalence models were preferred over H_5 in these four cases. Model H_4 was preferred for the comparison involving two-step addition and three-step application of the distributive property and the comparison of two-step addition with the five-step distributive property problem. Model H_3 was preferred for the comparison of two-step multiplication and two-step factoring. These cases provide additional evidence of boundary permeability.

Figure 1 summarizes both the within-domain and between-domain findings. The circles indicate domains. Ordering of tasks within domains and between domains is indicated by position in the vertical dimension. The long tube penetrating the two circles represents permeability in domain boundaries.

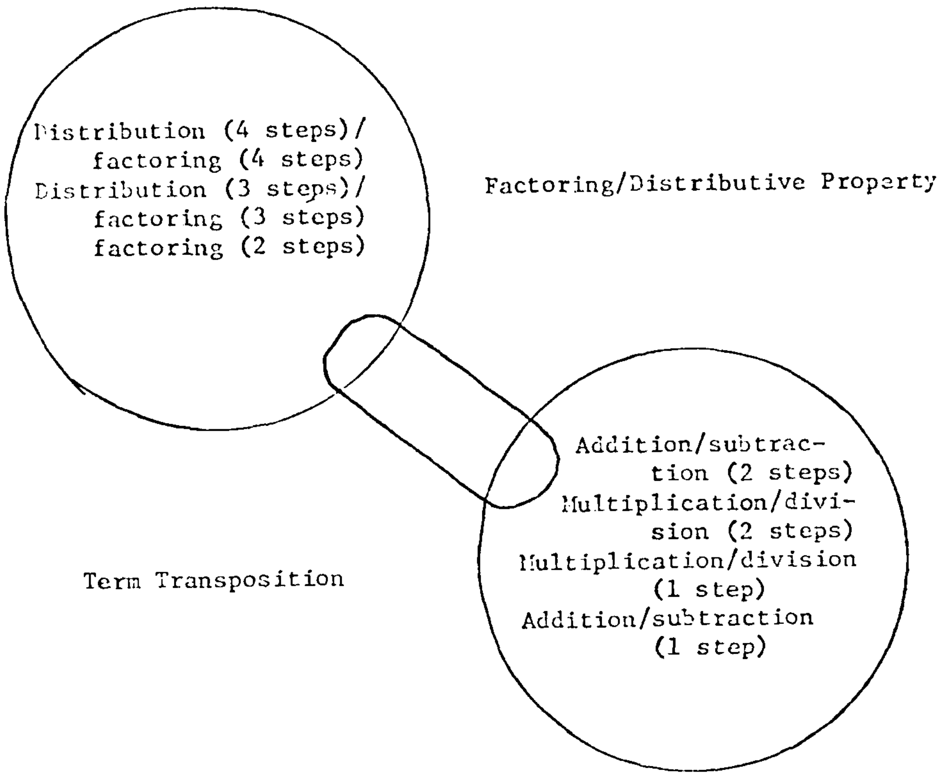

Figure 1.

Discussion

The results for task comparisons both within and between domains supported the major hypotheses advanced in the study. The within-domain findings are congruent with the view that algebra tasks representing a class of mathematical operations may be organized into homogeneous domains that involve asymmetrical equivalence relations. Moreover, as hypothesized asymmetrical equivalence is related to the number of steps required to achieve problem solution. The discovery of asymmetrical ordering raises questions about generalization and transfer within domains that may be important for instruction. For example, it is possible that instruction in a high difficulty task but also in generalization to low difficulty tasks. By contrast, instruction in a low difficulty task might not generalize directly to high difficulty problems. However, mastery of a low difficulty task could mediate positive transfer facilitating high difficulty task learning. Possibilities such as these call for research relating domain structure to generalization and transfer issues.

The unexpected finding that factoring and distributive property problems involving term and expression multiplication were in the same domain suggests that homogeneous domains may encompass rather broad classes of tasks. While it is true that factoring and multiplying an expression by a term are both regarded by mathematicians as applications of the distributive property, these tasks are nonetheless quite different in terms of the specific operations that they require. The fact that they were found to be in the same domain suggests that generalization of algebra skills may be very broad indeed. Research is needed to determine the breadth of generalization within domains.

The results for the between domain comparison support the hypothesis that ordered relations may exist between pairs of tasks in which one task is a component of the other. This finding linked to the within-domain results raises additional generalization and transfer questions with potentially important instructional implications. All of these relate to the question of how a student can best advance from a subordinate domain to a superordinate domain. For example, it would be of interest to know whether positive transfer would be significantly greater from a high difficulty subordinate domain task to a low difficulty superodinate domain task than from a high difficulty subordinate task to a high difficulty superordinate task.

The results with respect to boundary permeability raise additional questions regarding advancement from a subordinate to a superordinate domain. The findings suggest that permeability may exist and thereby raise the possibility of direct generalization between subordinate and superordinate domains. However, ambiguity in the permeability findings indicate the need for further research on the permeability phenomenon before conducting generalization studies. Permeability did not always occur in the manner hypothesized. In some cases it did take place as expected between the top level of the subordinate domain and the bottom level of the superordinate domain. However, in other instances it involved problems not adjacent to the domain boundary. This can be explained by the fact that to some extent permeability may be a function of unreliable responding. For example, a large number of individuals performing inconsistently on an hypothesized superordinate task would produce a buildup in the 32 cell that could mask the presence of an ordered relation. Examination of the observed response patterns in Table 9 suggests that some instances of apparent permeability may have resulted from high levels of inconsistent superordinate task responding. However, it is also true that the numbers in the 31 cell were generally smaller for task pairs close to the boundary between domains than for pairs far from the boundary. This suggests permeability. In order to resolve the permeability question, constant low levels of inconsistent superordinate task responding would be required. Further research is needed to study the relation of permeability to response inconsistency.

Second Year Research

The major focus of research for the second year was on the instructional validation of domains (Task 3) and hierarchies (Task 4). A different set of problem solving tasks was used because of observed ceiling effects for the tasks used in the first year. The tasks were too easy. Consequently, it was necessary to select a new set of algebra tasks and psychometrically validate these before proceding to the studies for Task 3 and Task 4. The second year research contained both outcome objectives and enabling objectives for the completion of Tasks 3 and 4.

Second Year Objectives

Outcome Objectives.

a. To validate item domains using an instructional validation technique.

b. To determine the extent to which psychometrically validated domains match domains validated using an instructional validation technique.

c. To validate hypothesized prerequisite relations and positive transfer in the training hierarchy using an instructional validation procedure.

d. To determine the congruence between psychometric and instructionally validated models assessing prerequisite relations and positive transfer in the training hierarchy.

e. To test the Gagné and Bassler (1963) random forgetting hypothesis by examining the congruence across learning and retention testing sessions of validated models describing prerequisite relations and positive transfer in the hierarchy.

The Enabling Objectives

a. To develop new tasks to meet the constraints of the data pool to be used. The simple algebra tasks employed in the first year were not appropriate for a college population.

b. To develop new models which represented different types of transitional classes, which represented the distinction between prerequisite ordering between domains and ordering by difficulty within a domain, and which incorporated response consistency into the contingency tables.

Models Used in the Second Year

New latent class models were developed for research in the second year. The models used in the first year did not allow for a specific test of Gagné's prerequisiteness hypothesis. Rather they tested for order on the basis of the assumption that under hierarchical ordering of two tasks there should be significantly more individuals who were masters of the superordinate task while being nonmasters of the subordinate task than who were masters of the superordinate task and in transition on the subordinate task. The models developed for the second research effort directly address the prerequisiteness question. They show that prerequisiteness alone is insufficient to determine ordering. However, they indicate that prerequisiteness can be used with other criteria to determine ordering. The models also provide explicit representation of various types of transition reflected in different kinds of response inconsistency. Different types of transition may occur when tasks are related, but not exactly the same. This is a typical state of affairs for most tasks of both academic and technical nature.

The first of the new models, labeled H_1 in the second year studies, describes the situation in which a domain is composed of only one task represented by equivalent items. Model H_1 includes three latent classes: A class of nonmasters who fail all items in the domain, a class of masters who pass all items, and a class of transitional individuals for whom the probability of a passing response is greater than zero and less than one. Since H_1 is designed to represent relations among equivalent items the probability of a passing response in the transitional class is assumed to be equal across items.

Model H_1 is congruent with the learning hierarchy model representation of hierarchical sequencing. The learning hierarchy approach assumes that each task in a hierarchy represents an equivalence class (Gagné, 1977). This is the assumption made under model H_1. Items grouped into an equivalence class form what Macready and Merwin (1973) have called a homogeneous item domain. Model H_1 offers one way to represent a domain of this kind (Bergan, in press; Bergan, Cancelli & Luiten, 1980).

The following restrictions are imposed on the conditional probabilities within the latent classes under model H_1:

$$\pi^{\bar{A}X}_{21} = \pi^{\bar{A}'X}_{21} = \pi^{\bar{B}X}_{21} = \pi^{\bar{B}'X}_{21} = 1$$

$$\pi^{\bar{A}X}_{12} = \pi^{\bar{A}'X}_{12} = \pi^{\bar{B}X}_{21} = \pi^{\bar{B}'X}_{21} = 1$$

$$\pi^{\bar{A}X}_{13} = \pi^{\bar{A}'X}_{13} = \pi^{\bar{B}X}_{13} = \pi^{\bar{B}'X}_{13} = 1$$

where $\pi^{\bar{A}X}_{21}$ is the probability of failing item A given membership in latent class 1 (nonmastery class), $\pi^{\bar{A}X}_{12}$ is the probability of passing item A given membership in latent class 2 (mastery class), and $\pi^{\bar{A}X}_{13}$ is the probability of passing item A given membership in latent class 3 (transition class). The other response probabilities are similarly defined.

Model H_2 represents a multiple task domain. Two tasks, each composed of two items are represented in the model. Model H_2 includes all three of the latent classes represented in H_1. In addition, it includes latent classes reflecting the assumption that the domain under examination contains similar but non-equivalent tasks. The assumption of related, but non-equivalent tasks suggests that there should be some tendency for performance on items representing the same task to be more similar than performance on items representing different tasks. This tendency implies that the transition between non-mastery and mastery will include cases of partially inconsistent performance. That is, there should be some tendency to perform consistently on one task while responding in an inconsistent fashion on the other. Four types of partial inconsistency could occur. These include non-mastery of task I accompanied by transitional responding for task II, mastery of task I with transitional responding for task II, non-mastery of task II linked to transitional responding on task I and mastery of task II coupled with transitional responding on task I. Each of these four types of partial inconsistency is represented by a latent class in model H_2.

The probability of the different classes of partially inconsistent performance is allowed to vary under model H_2. For example, the probability of performing task I items correctly while responding inconsistently on task II items could be higher than the probability of performing task II items correctly while responding inconsistently on task I items. This type of variability reflects differences in task difficulty during the transition between non-mastery and mastery.

Assumptions about partial inconsistency are reflected in the following restrictions on transition classes:

$$\pi^{\bar{A}X}_{14} = \pi^{\bar{A}'X}_{24}, \ \pi^{\bar{B}X}_{24} = \pi^{\bar{B}'X}_{24} = 1$$

$$\pi^{\bar{A}X}_{15} = \pi^{\bar{A}'X}_{15} = 1, \ \pi^{\bar{B}X}_{15} = \pi^{\bar{B}'X}_{25}$$

$$\pi^{\bar{A}X}_{26} = \pi^{\bar{A}'X}_{26} = 1, \ \pi^{\bar{B}X}_{16} = \pi^{\bar{B}'X}_{26}$$

$$\pi^{\bar{A}X}_{17} = \pi^{\bar{A}'X}_{27}, \ \pi^{\bar{B}X}_{17} = \pi^{\bar{B}'X}_{17} = 1$$

Note that in each of the above transition classes the restrictions for the items representing one task indicate consistent responding and the restrictions for the other task indicate inconsistent responding. For instance, in latent class 5, items A and A' are restricted to occur at level 1. By contrast, items B and B' are restricted to represent inconsistent responding.

Model H_3 is exactly the same as model H_2 in all respects except that it imposes restrictions on the latent classes that represent the case in which tasks within a domain are assumed to be of the same difficulty level for transitional individuals. The assumption of equal difficulty is reflected with the following restrictions on the latent classes in the model:

$$\pi^X_4 = \pi^X_6, \ \pi^X_5 = \pi^X_7$$

where π^X_4 is the probability of occurrence of latent class 4 and the others are similarly defined. The first restriction depicts the assumption that the probability of consistently failing task II while responding inconsistently to task I will be equal to the probability of consistently failing task I and responding inconsistently to task II. The second restriction handles the same assumption with respect to passing the items.

Models H_4 and H_5 are designed to represent the hierarchical ordering of skills. The learning hierarchy model has been highly influenctial in providing a basis for establishing hierarchical ordering. The sole criterion used to determine hierarchical ordering within the learning hierarchy model is the prerequisiteness criterion (Gagné, 1962, 1977; White & Clark, 1973). This criterion states that one skill is prerequisite to another if, given suitable allowance for measurement error, no one has mastered the superordinate skill without also having acquired the subordinate skill (White & Clark, 1973). This criterion is not sufficient to establish an ordered relation between skills.

The difficulty with the prerequisite skills criterion is that it does not distinguish between equivalence and hierarchical ordering. For example, consider the case of two identical tasks. If the tasks are truly identical, individuals who have mastered one should also evidence mastery of the other. Likewise, individuals who are nonmasters of one should display nonmastery of the other. Finally, individuals in transition between nonmastery and mastery should perform inconsistently on both tasks. Given appropriate allowance for measurement error, there should be no one who displays mastery on one task and nonmastery on the other. As this example shows, equivalence is a special case of prerequisiteness in which the prerequisiteness criterion can be applied regardless of which task is assumed to be superordinate.

Models H_4 and H_5, used to represent hierarchical ordering in the present research, assume that the central criterion for establishing ordering should be the existence of a class of individuals who have mastered the hypothesized subordinate skill and at the same time are nonmasters of the hypothesized superordinate skill. The prerequisite criterion is used as an additional tool in establishing ordered relations. Thus, two kinds of ordered relations are assumed. One requires only that there be a group of individuals who have mastered a subordinate task and at the same time are nonmasters of the superordinate task. The second also imposes the prerequisiteness criterion.

Model H_4, the ordered relation model, includes all the latent classes in model H_2 plus an additional class. The additional class involves individuals who are masters of the subordinate task and nonmasters of the superordinate task. This class provides the fundamental criterion for determining ordered relations and the following restrictions are imposed upon it:

$$\pi^{\bar{A}X}_{18} = \pi^{\bar{A}'X}_{18} = \pi^{\bar{B}X}_{28} = \pi^{\bar{B}'X}_{28} = 1$$

where $\pi^{\bar{A}X}_{18}$ is the probability of passing item A given membership in latent class 8 and $\pi^{\bar{B}X}_{28}$ is the probability of failing item B given membership in latent class 8. The other conditional probabilities are similarly defined

Model H_4 assumes ordering, but it is not entirely consistent with the assumption of prerequisiteness. For example, H_4 includes transition individuals who may respond as nonmasters of the subordinate task but who respond inconsistently on the superordinate task.

Model H_5 represents a prerequisitely ordered relation between tasks. It reflects ordering and the notion of prerequisiteness by ruling out the transition classes inconsistent with the hypothesis of ordering. Model H_5 is like H_4 except for the exclusion of two transitional classes. One of these classes depicts inconsistent superordinate task performance accompanied by nonmastery of the subordinate task. The other represents mastery of the superordinate task accompanied by inconsistent performance on the subordinate task.

Model H_5 can be used as a basis for establishing the boundary between hierarchically ordered domains (Bergan, Note 4). The use of H_5 in boundary definition ensures that there will be a significant number of individuals who have mastered the tasks in the subordinate domain without having mastered the tasks in the superodinate domain. In addition it assures conditions compatible with Gagné's (1962) prerequisiteness criterion.

The latent class models described above were used in the studies described below to represent the domain hypothesis and prerequisite ordering hypothesis. Structural equation models (Bergan, 1980) were used to examine the positive transfer hypothesis. The models employed in the research made use of a modified path analysis technique developed by Goodman (1973). This technique is outlined in Appendix C in conjunction with the description of latent class techniques.

STUDY I

The first study conducted in the investigation was a psychometric validation study addressing the domain, prerequisite ordering, and positive effects hypotheses. Five algebra problem solving tasks each represented by two identical items were used in the study. Two of the tasks involved the solution of quadratic equations while the other three required the solution of cubic equations. The task demands were varied both for the two quadratic problems and for the three cubic problems. One of the quadratic problems imposed greater search demands (Newell & Simon, 1972) on the problem solver than the other. That is, the number of possible combinations of numbers that had to be considered to arrive at a solution was greater for one problem than for the other. The cubic equations incorporated similar variations so that the three cubic problems were ordered in terms of search requirements.

It was hypothesized that the two quadratic tasks would form one domain and that the three cubic tasks would form a second domain. Because of the variations in task demands within domains, it was assumed that there would be differences in task difficulty within domains. It was further assumed that the two domains would be ordered hierarchically . That is, each of the tasks in the quadratic domain would be prerequisitely ordered with respect to each of the tasks in the cubic domain.

Method

Subjects. The subjects were 203 university students who were taking introductory psychology courses and who volunteered to participate in the investigation. There were approximately equal numbers of males and females representing a wide range of educational backgrounds with respect to algebra problem solving. Some had had many courses involving algebra skills. Others had had only one introductory course in highschool.

Tasks. The item form approach (Hively, 1974, Hively, Patterson & Page, 1968) was used to select the two quadratic and three cubic equation problems used in the study. The use of the item form technique ensured that each of the problems selected would represent a well-defined class of items. The two quadratic tasks selected were: $X^2 + 6X + 5 = 0$ and $3X^2 + 3X - 18 = 0$. These were labeled Q_1 and Q_2, respectively. The roots representing the solutions to these quations can be obtained by factoring. The first quadratic can be factored into the expressions $(X + 5)(X + 1) = 0$. The second can be factored into the expressions $(3X + 9)(X - 2) = 0$. Note that in the first problem there is only one set of whole numbers that can be multiplied together to produce the 5 constituting the third term in the equation, namely 5 X 1 . On the other hand there are several ways to produce the -18 constituting the third term in the second equation (e.g., -6 x 3, -3 x 6, 9X - 2, -2 x 9). Many more combinations of numbers have to be considered to factor Q_2 than to factor Q_1. This illustrates the fact that the search requirements attendant to solving Q_2 are greater than the search requirements associated with solving Q_1.

The three cubic equations were $4X^3 - 20X^2 + 20X = 0$, $3X^3 + 7X^2 + 2X = 0$, and $24X^3 + 28X^2 + 8X = 0$. These were labeled C_1, C_2, C_3 respectively. The cubic equations varied in search requirements with C_1 being the least demanding and C_3 being the most demanding.

Procedures. Two identical problems, for each of the five above tasks, were presented to the subjects together with forty-seven other algebra problems. The problems were arranged in a single random sequence in test booklets. Pairs of identical items were used to take into account inconsistency in responding. The purpose of the other algebra tasks was two-fold. Since identical items were used, the other algebra tasks served to reduce the likelihood of individuals remembering answers to problems they may have already completed. In addition, the other tasks provided data for further research on the hieraichical arrangement of math skills. All items were scored dichotomously, either correct or incorrect. A correct answer was one in which all the roots to an equation were identified by the subjects.

The administration of the test was carried out in groups ranging in size from five to ten. The participants were told the purpose of the study and instructed to solve for all possible values of X in the problems. In addition, the subjects were told to not look back at any previously done work. During the course of the testing, the experimenter monitored the trainees' performance to ensure that all the items were attempted by the subjects and that all directions were being followed.

Results

The five latent class models described in the preceeding section were used to test the domain hypothesis and prerequisite ordering hypothesis. Model testing was conducted for all possible cross-classifications of quadratic and cubic equations. The observed response patterns for these cross-classifications are given in Table 10. Note the relatively large numbers of individuals in the quadratic-cubic cross-classifications who passed both quadratic items and failed both cubic items. This suggests a hierarchical ordering. Likewise, note the extremely small number of individuals who respond correctly more often on a superordinate than a subordinate task. This suggests prerequisiteness.

Table 11 presents the results of model testing for each of the cross-classifications of quadratic and cubic equations. Asterisks indicate the "preferred" model for each task set examined. Preferred models were arrived at by making statistical comparisons between the various models tested (Bishop, Feinberg, & Holland, 1975; Goodman, 1974). The likelihood ratio statistic was used in all the studies because it can be partitioned exactly.

For example, consider the chi-square tests for the $Q_1 - Q_2$ cross-classification displayed in Table 11. Model H_1 yields an X^2_L of 29.07 with 12 degrees of freedom. This model does not fit the data ($p < .01$). However, it is hierarchically related (Goodman, 1974) to H_2. That is, model H_2 contains all of the characteristics of H_1 plus eight more reflecting the four partially inconsistent latent classes of model H_2. Inclusion of these latent classes reduces the available degrees of freedom by 8. Because they are hierarchical, H_2 and H_1 can be compared statistically. The X^2_L of H_2 can be subtracted from the X^2_L for H_1. The result is an X^2_L of 22.84 with 12 - 8 = 4 degrees of freedom, which is significant well beyond the .001 level. Thus, model H_2 improves significantly on H_1 and provides an adequate fit for the data.

The next step in the model comparison process would be to compare model H_2 with model H_4 in order to examine the nature of the ordering relationship between two tasks. Model H_4 contains all the classes of H_2 plus one additional class reflecting ordering. Model H_4 has three degrees of freedom. Model H_4 and H_3 are hierarchically related. The subtraction of the two X^2_L values yields a X^2_L of .30 with 1 degree of freedom ($p > .5$) which is not significant.

Since model H_2 improves the fit over H_1 and is a more parsimonious representation of the data than offered under model H_4, it is a candidate for adoption as the preferred model. However, before a final decision can be made, it is necessary to consider H_3. Model H_3 fits the data adequately and is hierarchically related to H_2. Model H_3 has two additional characteristics imposed upon the latent classes that are common to models H_2 and H_3. These restrictions are designed to represent tasks of the same difficulty level. These two restrictions afford two additional degrees of freedom for model H_3. The subtraction of the X^2_L for model H_2 from the X^2_L for H_3 yields a X^2_L of 1.58 with 2 degrees of freedom ($p > .5$). Since H_2 does not significantly improve the fit afforded by H_3, model H_3 is the preferred model.

The preferred models displayed in Table 2 demonstrated congruence with the hypothesis advanced earlier. The data for the cross-classification of Q_1 and Q_2 was best fit under model H_3. This suggested that Q_1 and Q_2 were in the same domain and of equal difficulty. Although the quadratic items did represent one domain as hypothesized, it was expected that Q_1 and Q_2 would exhibit some difference in difficulty given the hypothesized search requirements characteristic of Q_1 and Q_2.

The model comparisons also revealed that a single domain existed for C1, C2, and C3. The preferred model for the C1=C2 task set was H_1, suggesting that C1 and C2 were essentially one task for the individuals. The C2-C3 cross-classification was best fit under model H_2. This indicated that these two items, although in the same domain, were different in difficulty level. The comparison of C1 and C3 manifested a greater difference in difficulty level than found for the C2-C3 cross-classification. The preferred model was H_4, indicating ordering but not prerequisiteness. Although C1-C3 were represented by one domain, a significant ordering by difficulty relationship was present.

The general structure to the cubic data was as hypothesized. One domain was realized and the variations in task difficulty that were found were expected. C1, the least demanding of the cubic equations, was shown to be much less difficult for subjects than C3, the most demanding of the three cubics. C2, the moderately demanding task, was less difficult than C3. The only contrary finding was the result that C1 and C2, two items with different amounts of hypothesized search requirements, were essentially the same task for individuals.

The comparisons of the quadratics with the cubics suggested that with one exception the quadratics and the cubics were in different domains. This was illustrated through the cross-classifications of Q1-C2, Q1-C1, Q1-C3, Q2-C2, and Q1-C3. The preferred model for the task sets was H_5, which suggested a prerequisite relation between these items and a hierarchical relationship between the quadratic and cubic domains. This was hypothesized to be the relationship between these two general types of tasks.

However, some overlap between these two domains was identified. For the comparison of Q2 and C1, the preferred model was H_2, which suggested that the two items were not equally difficult but nonetheless in the same domain. This indicated that the two domains were not entirely disassociated, but that some permeability between the two domains existed.

The last hypothesis tested examined the effects of subordinate skill levels on the acquisition of superordinate skills. Goodman's (1973) modified path analysis approach was used to assess positive effects for tasks representing both the quadratic and the cubic domains. The Q1, C2, and C3 items were selected because the tasks represented by these items were used in the instructional validation of positive effects conducted in study 3. For the case of two variables, the effect of subordinate competency on superordinate competency can be tested by the usual chi-square test of independence (Fienberg, 1977).

The test of the effect of Q1 on C2 resulted in a X^2_L value of 72.47 with 1 degree of freedom ($p = 0.00$). In a similar test, the effect of Q1 on C3 yielded a X^2_L value of 51.75 with 1 degree of freedom ($p = 0.00$). This supports the hypothesis that Q1 competency does effect the competency level achieved in C2 and C3.

Discussion

The results revealing the hierarchical ordering of quadratic and cubic asks suggest that these tasks must differ in certain fundamental ways. There are two important differences between the quadratic and cubic tasks. One is that the cubic equations require the identification of three roots whereas the quadratic equations call for the identification of only two roots. The second is that each cubic equation requires as a first step that an X be factored from the three term expression on the left side of the equation. Factoring is not required in the quadratic case. Indeed, any attempt to factor an X in the case of a quadratic problem would tend to make it impossible to solve the problem (Bundy, Note 5; Carry, Lewis, & Bernard, Note 6). Thus, what was a required initial step in the cubic case was an incorrect step in the quadratic case.

The results within domains contained unexpected findings. In particular, the assumption that variations in search requirements would produce variations in task difficulty within domains was supported only for the $C_1 - C_3$, $C_2 - C_3$ task pairs. In all other cases the tasks under study within domains were shown to reflect equal difficulty levels.

The within domain findings suggest that examinees were able to reduce the search demands associated with the various problems to an extent that minimized variations in task difficulty. Search requirements for the problems could be reduced in two ways. One involved the use of the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This formula eliminates the search process required in the factoring approach to the solution of the equations. An informal analysis of examinee protocols revealed that a small number of examinees did use the quadratic formula to good advantage. However, the vast majority used the factoring approach. The observed preference for factoring is consistent with findings reported by Carry, Lewis and Bernard (Note 6). Those who attempted to factor the left sides of the equations, for example, into expressions such as $3(X + 3)(X - 2) = 0$ could reduce search requirements by such means as initially factoring a number from the three terms in the left side of the equation. This type of strategy was used by the vast majority of examinees who were successful in solving the problems.

It was anticipated that examinees would use different strategies to reduce search requirements in the problems. What was not expected is that for the most part they would be equally effective in applying the appropriate strategies to the various problems. Yet, with the exception of the C_3 problem, this was the case.

Although the tasks within domains tended to be of equal difficulty, for the most part the items did not form an equivalence class of the sort that might be expected under the learning hierarchy model. Model H_1 asserting equivalence among items fit the data for only one task set, $C_1 - C_2$. The analyses for all of the other task pairs supported the assumption that each task formed a subclass within the domain to which it belonged. Tasks C_1 and C_2 are not apparently any more similar than any other of the task pairs, and it is not certain why they should be well represented by model H_1 when H_1 was not preferred for the other task pairs.

The results for the structural analysis examining the positive effects hypothesis require brief comment. Given the existence of two domains, the finding the Q_1 affected both C_2 and C_3 is exactly what should be expected. By contrast if C_2 and C_3 were in two separate hierarchically ordered domains, there should have been no direct effect of Q_1 on C_3 (Bergan, 1980). Rather, Q_1 would exert its influence on C_3 indirectly through its effects on C_2.

STUDY 2

Study 2 was an instructional validation study designed to examine the domain hypotheses. The major purpose of the study was to determine the congruence of psychometric and instructional validation findings with respect to the domain memberhips of quadratic and cubic tasks.

The domain hypothesis takes on a somewhat different meaning under the instructional validation paradigm than it has under the psychometric paradigm. In the psychometric validation case, tasks may be said to be in the same domain if data can be adequately represented by mastery, nonmastery and transition latent classes, each of which is characterized by a certain type of response pattern. The same latent classes may be used in domain validation under instructional conditions, but they may take on a different interpretation in the instructional validation case. More specifically, some of the latent classes indicate generalization under the instructional paradigm whereas they do not necessarily imply generalization under the psychometric paradigm. For example, consider the case in which a group of individuals is trained on quadratic task I and then tested on that task and quadratic task II. Under these conditions the mastery class represented by correct performance on both tasks indicates a group of individuals who have generalized task I skill to task II. Under psychometric validation conditions mastery indicates merely that there is a group of individuals who respond correctly on both tasks.

It is not just the mastery class that implies generalization under an instructional validation paradigm. Each of the various possible transition classes also implies a degree of generalization. It is only the nonmastery class that signifies a lack of generalization.

The process of determining the congruence between psychometric and instructional validation can be conceptualized as an investigation of whether or not models representing the tendency to respond in the same way on different tasks are congruent with findings for models representing generalization across tasks. In the process of determining the congruence of psychometric and instructional techniques, study 2 investigated generalization within the quadratic and cubic domains identified in study 1. It examined the question of whether or not individuals trained on one set of Q_1 problems would generalize what they had learned to other Q_1 problems and to Q_2 problems. Likewise, it investigated the extent to which individuals trained on a set of Q_2 problems would generalize their learning to other Q_2 problems and to Q_1 problems. It examined the generalization issue in an analogous fashion for the C_1 and C_2 tasks from the cubic domain.

The issue of generalization is of interest not only within domains, but also between hierarchically ordered domains. Insofar as hierarchical ordering implies the existence of a class of individuals who are masters of the skills in the subordinate domain and nonmasters of the skills in the superordinate domain, it is reasonable to assume that generalization from a subordinate domain task to a superordinate domain task should be minimal. However, generalization should be substantial from a superordinate task to a subordinate task. Gagné (1973) has hypothesized that an individual who learns a superordinate skill without having received instruction in the related skill will in the course of mastering the superodinate skill acquire the prerequisite skill. This hypothesis implies downward generalization from the superordinate skill to the subordinate skill. Cotton, Gallagher and Marshal (1977) discuss this type of downward generalization and point out that it has not been studied.

Study 2 examined both upward and downward generalization between the quadratic and cubic domains identified in Study 1. It investigated the extent to which individuals trained on Q_1 or Q_2 problems generalized to C_1 and C_2 problems and it examined the extent to which individuals trained on C_1 or C_2 problems generalized to Q_1 and Q_2 problems.

Method

Subjects. The subjects were 325 volunteers from the University of Arizona enrolled in an Introductory to Educational Psychology class who represented a wide range of skill levels insolving algebra problems.

Tasks. Two tasks from the quadratic domain (Q_1 and Q_2) and two tasks from the cubic domain (C_1 and C_2) were used in the study. All items were generated using the item forms and item form shells used in Study 1. Different items were generated for each training phase and each testing phase of the study.

Procedures. A pretest was first given to the 325 volunteers in groups of approximately twenty in size. It consisted of two problems for each of the above four tasks in a random sequence. Identical problems were not used because only eight problems were presented to the subjects. However, the pairs of items were isomorphic since they were generated from the same item forms and item form shells. The administration of the pretest involved the same directions and monitoring activities used in Study 1. Since Study 2 was to involve instruction in the four above math problems, it was necessary to identify those trainees who could not correctly solve any of the problems on the pretest. Failing a problem was defined as not correctly solving for all of the values of X in the equation. Of the 325 volunteers, 246 met the criterion of failing all the items on the pretest. Those who qualified for the training portion of the study were randomly assigned to one of four training conditions involving instruction in Q_1, Q_2, C_1, and C_2.

Training involved presenting the subjects with a problem and then modeling feedback demonstrating a sequence of steps leading to problem solution. The demonstration included verbal descriptions of rules and strategies applied in the problem solving. The rules and strategies included determing the number of roots, factoring and eliminating common elements, simplifying equations into factored expressions, obtaining roots, and finally checking the roots. These steps are illustrated in Table 12, however, in a more abbreviated form than used in the study. Both the problems and the demonstration feedback were presented in written form in a manner analogous to that used in programmed instruction. The training materials used in Studies 2 and 3 are provided in Appendix B.

Three trials were given. The amount of feedback was reduced systematically over the three trials. On trial 1, feedback included detailed illustrations of all operations used in achieving a problem solution. On trial 2, some of the operations required to carry out a particular step in problem solving were not spelled out. Specifically, the steps required to factor the equation into expressions were reduced in trial 2. Trial 3 feedback contained only a sequence of equations. No verbal descriptions of steps were included. Table 13 illustrates a portion of the type of feedback provided during each of the three trials.

To attain a correct score, all solutions to an equation had to be identified by an individual.

Results

The five latent class models used in Study 1 were employed to examine the congruence of instructionally validated and psychometrically validated domains. As in Study 1, statistical comparisons were made between models to arrive at a preferred model. The model testing was conducted for only those cross classifications of the quadratics and cubics that involved the specific task on which subjects were trained. For example, for those subjects trained in Q_1, only the comparisons of $Q_1 - Q_2$, $Q_1 - C_1$, and $Q_1 - C_2$ were made.

The observed response patterns for all the various cross-classifications of tasks are given in Table 14. The cross-classifications for the groups trained in the two quadratic tasks suggest a hierarchical ordering between the quadratic and cubic tasks. This is evidenced by the large number of subjects who passed both quadratic items but failed both cubic items. A prerequisite relationship is further indicated by the extremely small number of trainees responding correctly more on the superordinate task than the subordinate task. This finding is highlighted by the very small numbers representing mastery of both the quadratcis and cubics following training in Q_1 or Q_2.

On the other hand, for those subjects trained in C_1 or C_2, there were significantly more trainees who mastered both the quadratic and cubic items. This suggests downward generalization from the cubic task to the quadratic task. It is apparent that the mastery and nonmastery cells in the table, the 1111 cell and the 2222 cell , account for the vast majority of the subjects' responses. This suggests that trainees who learned the cubic task also learned how to solve the quadratic tasks. Few subjects actually learned the cubic task and failed the quadratic task when trained in C_1 or C_2.

The results of the model testing are presented in Table 15. In the case of subjects trained in Q_1 or Q_2 the quadratic items were found to prerequisitely ordered with the cubic items. Model H_5, indicating prerequisite ordering, was the preferred model for the $Q_1 - C_1$, $Q_1 - C_2$, $Q_2 - C_1$, and $Q_2 - C_2$ task sets. The analyses for the $Q_1 - Q_2$ task set for these same training groups indicated that Q_1 and Q_2 were of the same domain. For subjects trained in Q_2, the preferred model was H_1, a model suggesting that Q_1 and Q_2 were essentially the same task for individuals. The preferred model for the $Q_1 - Q_2$ cross-classifications of the group trained in Q_1 was H_2. Model H_2 offered the interpretation that although the two items were in the same domain they were not equally difficult.

The Q_1 and Q_2 training conditions afforded the opportunity to examine upward generalization. Upward generalization did not occur. This is apparent from the observed response patterns. No statistical test was needed to verify the lack of generalization. In cases of training under either Q_1 or Q_2 there were never more than two people who responded correctly to a cubic item.

The cubic items, like the quadratic items, were shown to be in the same domain. This is indicated by the $C_2 - C_1$ analyses for subjects trained in either C_1 or C_2. The preferred model for both groups was H_2, a model suggesting the existence of a single domain with items which were of different difficulty levels.

Training under C_1 and C_2 afforded the opportunity to examine downward generalization. The analysis for the various cross-classifications of cubic and quadratic tasks indicated that such generalization did occur. However, it did not occur in the same way under C_2 training as it did under C_1 training. For subjects trained in C_1 problems, model H_1 indicating task equivalence was preferred. This suggested that if C_1 was learned, Q_1 and Q_2 were also learned and if C_1 was not learned, Q_1 and Q_2 were not learned.

By contrast, for subjects trained under C_2, H_4 indicating an ordered relation was the preferred model. This indicated that when C_2 was learned, Q_1 and Q_2 were also learned. However, it also suggested that C_2 training had a beneficial effect on Q_1 and Q_2 learning even when C_2 was not learned.

Discussion

The results for this study support the general hypothesis that domains validated psychometrically can be expected to be congruent with domains validated instructionally. The analyses for both quadratic equations and cubic equations revealed complete congruence in domain assignment. These results support the assumption that domain membership determined psychometrically is congruent with generalization of skills occurring in the course of skill learning. The occurrence of generalization has instructional significance in that it indicates that instructors may provide instruction in one skill in a skill domain and anticipate that a significant number of learners will generalize that skill to the performance of other skills in the domain.

The occurrence of generalization is both a desired and an expected outcome in educational environments, and it is not surprising that it occurred. What is particularly important to point out is that psychometrically validated domains were useful in predicting the extent of occurrence of generalization. Instruction in the quadratic domain produced generalization within that domain. Yet, there was no generalization to the cubic domain. By contrast, as predicted, instruction in the cubic domain produced downward generalization to quadratic tasks. As these results show, the trainee who can master a superordinate task without receiving instruction in the subordinate task derives a double benefit. Congruent with Gagné's (1973) expectations, the trainee may acquire not only the superordinate skill but also the subordinate skill as a result of receiving superordinate skill instructions. A corrolary to this result is that even though a learner may not master the superordinate skill, superordinate skill instruction may eventuate in subordinate skill mastery. This state of affairs occurred in the case of trainees who received instruction on C_2 problems. The preferred model for both the Q_1-C_2 and Q_2-C_2 task sets given C_2 training was H_5 indicating prerequisite ordering. Model H_5 includes a class of individuals who are nonmasters of the superodinate skill and masters of the subordinate skill indicating that some learners mastered Q_1 and Q_2 tasks while being nonmasters of C_2 under instruction in C_2 tasks.

The above generalization findings suggest the advisability of targeting instruction markedly ahead of the learner's current level of skill rather than following the well accepted maximum of taking the learner from his/her current level in a step by step fashion to higher levels of achievement. Presumably, there is some limit to how far ahead one can skip before reaching a situation in which little or no learning occurs. The feasibility of skipping domains may also vary based on the individual characteristics of learners. Factors affecting learning when subordinate domains are skipped is a topic in need of much research.

Although the findings for Study 2 were generally congruent with the findings reported for Study 1, there were some interesting differences between Study 1 and Study 2 results. In the case of the two quadratic items under psychometric validation, model H_3 was preferred, indicating separate tasks of equal difficulty. Under instructional validation, the preferred model for the Q_1- Q_2 task set was H_1 under Q_2 training and H_2 under Q_1 training. This indicated that when training was provided for Q_1, the task hypothesized to be the least difficult, Q_2, was shown to be more difficult than Q_1. On the other hand, when training was provided for the more difficult Q_2 task, the two tasks were shown to be equivalent. This suggests complete downward generalization and partial upward generalization within the quadratic domain.

For the cubic domain, the results were somewhat different. Under psychometric validation model H_1 was preferred for the C_1- C_2 task set. However, under instructional validation, H_2 was preferred under both C_1 and C_2 training. Thus, training on the most difficult task did not produce task equivalence (i.e. complete downward generalization) in the cubic domain as it did in the quadratic domain.

There are differences in task relations for the cubic and quadratic domains that could account for the lack of congruence in downward generalization for the quadratic and cubic domains. In the case of the quadratic domain factoring a number from the expression on the left side of the equation in a Q_2 problem led to an expression identical to the left side of a Q_1 problem. Thus, a Q_2 problem could be described as being exactly the same as a Q_1 problem except that it required an additional step. This was not the case for the cubic problems. Problems of the C_2 type could not be related to C_1 problems by performing an additional step. Thus, it is not too surprising that the cubic problems varied in difficulty regardless of whether training occurred under C_1 or C_2.

The fact that quadratic and cubic problems varied in difficulty under instructional validation, but not under psychometric validation, requires comment. The psychometric validation study reflected students development of algebra problem solving skills over a long time span. In the course of development, trainees have the opportunity to acquire a large repertory of problem solving strategies. By contrast, under instructional validation conditions the opportunities for strategy acquisition are necessarily limited. The range of problems to which trainees can be exposed is small and the time allotted for constructive thought about the problems presented is minimal. Thus, it is reasonable that differences in task difficulty will appear to be more pronounced under instructional validation than under psychometric validation.

A final discrepant result between the psychometric and instructional validation that calls for discussion involves domain permeability. Permeability occurred under psychometric validation, but not under instructional validation. This may have happened for the same reason as that advanced with respect to the discrepant findings discussed in the last paragraph. Differences in task difficulty may be more pronounced under instructional validation than under psychometric validation.

STUDY 3

Study 3 was an instructional validation study. One purpose of the investigation was to determine the congruence of psychometrically validated and instructionally validated prerequisite orderings. As indicated earlier, previous research (Gagné & Bassler, 1963; White, 1976; White & Gagné, 1978) raised the question of whether or not psychometrically validated hierarchies can be expected to yield results congruent with instructionally validated hierarchies. A related question had to do with whether or not skills may be forgotten in a different order than the order in which they are learned. Study 3 investigated both of these questions.

A second purpose of the investigation was to determine the congruence of positive effects established psychometrically and positive effects established instructionally. This involved determining whether or not the same structural model was preferred under instructional validation as under psychometric validation.

A third goal of the study was to determine the nature of positive effects validated under instructional conditions. The meaning of the positive effects hypothesis may be different under psychometric and instructional validation procedures. In the case of psychometric validation, to say that skill Q_1 has a positive effect on skill C_2 implies that skill Q_1 has a causal influence on skill C_2. Three conditions have been advanced as necessary for hypothesizing a causal relation in the structural equation literature (Bergan, in press; Heise, 1975). One is a theoretical justification for the hypohtesized causal relation. The second is assurance that the causal variable occurs in time either prior to or simultaneously with the variable that it is assumed to effect and the third is empirical evidence of a relationship between the two variables. Psychometric validation can satisfy these three conditions. An appropriate theoretical model can be provided asserting a causal effect between skills. Validation of prerequisite ordering can assure that the causal variable occurs before or simultaneously with the variable that it is presumed to effect. Finally, application of structural equation techniques can determine the existence of a relationship between the variables of interest.

Although the psychometric technique satisfies the conditions for determing of causal relation, there remains some ambiguity in the interpretation of the term effect under the psychometric validation approach. In particular to say that one skill affects another under psychometric validation may imply that mastery of the first skill influences the performance of the second skill in the absence of training with respect to the second skill or it may mean that mastery of the first skill influences the learning of the second skill. The former of these cases implies generalization whereas the latter implies transfer. Instructional validation also may involve generalization or transfer. However, in the case of instructional validation, it is possible to separate generalization effects from transfer effects.

Study 3 examined the question of whether or not a positive effect of a subordinate skill on a superordinate skill implied generalization or transfer or both. In addition , it looked at the question of whether or not a lower order skill within a domain affected generalization and/or transfer to a higher order skill within the domain.

Method

Subjects. One hundred and seventy-two subjects in an intorductory psychology course volunteered for this study.

Tasks. One task from the quadratic domain (Q_1) and two tasks from the cubic domain (C_2 and C_3) were used in this study. All items were generated using the item forms and item form shells used in study 1. Different items were generated for each training and testing phase of the study.

Procedures. All the trainees first demonstrated their inability to correctly solve all the problems targeted for investigation on a pretest. To correctly respond to an item, an individual had to identify all the solutions to a problem. The pretest consisted of two problems for each of the above three tasks presented in a random sequence. Items were generated using the item form approach described previously. The administration of the pretest involved the same directions and monitoring activities used in Study 1 and Study 2.

Following the pretest, 172 trainees received introduction in Q_1, C_2 and C_3 in that order. The instruction was given in groups ranging in size from three to eight, and was of the exact format used in Study 2. Following each instructional phase the subjects took a six item test. These tests always contained two items from each of the three tasks (Q_1, C_2, and C_3). After undergoing the three training phases and completing the three tests, the 172 trainees were asked to return one week later to take a final six item retention test. Of the 172 original subjects, 167 returned for this test. In all cases, items were scored dichotomously, either correct or incorrect.

Results

The five latent class models used in studies 1 and 2 were employed to assess the congruence of the prerequisite relations validated psychometrically and instructionally. Statistical comparisons were made between the models to arrive at a preferred model in the same manner used in studies 1 and 2. The model testing was conducted for all possible cross-classifications between Q_1, C_2, and C_3 on each test taken by the subjects with one exception. This was the case of the $C_2 - C_3$ task set on the test following Q_1 instruction. This specific cross-classification was of no interest since no exposure to C_2 or C_3 had occurred.

The observed response patterns for all the various cross-classifications are given in Table 16. The cross-classifications for the subjects following training in Q_1 suggest the by now familiar hierarchical ordering between the quadratic and cubic tasks. Note the large numbers of trainees passing the two Q_1 items and failing both the C_2 and C_3 items. The cross-classifications of the quadratics with the cubics following training in the cubics shows a marked change in these numbers. Now the majority of individuals fall into the mastery and nonmastery latent classes. The $C_2 - C_3$ cross-classifications displayed few trainees passing one task but not the other. This suggests that C_2 and C_3 are of the same domain.

The results of the model testing are presented in Table 17. Following training in Q_1, the model testing revealed that the quadratics and the cubics were of different prerequisitely ordered domains. This was indicated by model H_5 being the preferred model. The same relationships held following training in both C_2 and C_3. The $C_2 - C_3$ cross-classification following training in the cubics suggests that the two tasks were in the same domain but of different difficulty levels. This is indicated by model H_2 being the preferred model. It is apparent that the prerequisite relations discovered in Study 1 were validated instructionally.

The findings from the test taken one week later are also presented in Table 15. The chi squares and the preferred model for the two quadratic and cubic cross-classifications indicate that the prerequisite relations between these two tasks was not lost. Quadratic and cubic tasks form two hierarchically ordered domains. The one anomaly that occurred in this study resulted from the analysis of C_2 and C_3 on the retention test. From Table 14, this cross-classification suggested that C_2 and C_3 were hierarchically arranged in two domains. This is indicated by the preferred model H_5.

Two path analyses were conducted. The first examined generalization and transfer effects involving Q_1 and C_2 skills. The Q_1 skill was examined immediately following Q_1 training. By contrast, the C_2 skill was studied at two points in time. Time 1 occurred immediately after Q_1 training and time 2 followed C_2 training. The analysis also examined the generalization of Q_1 to C_2 at time 1 and the generalization of C_2 at time 1 to C_2 at time 2. Generalization was studied to facilitate the separation of generalization effects from transfer effects. The association between Q_1 and C_2 can be conceived of in terms of two components, a generalization component and a transfer component. In the path analysis the transfer component was represented by the direct effect of Q_1 on C_2 at time 2. The generalization component was represented by the indirect effect of Q_1 on C_2 at time 2. This indirect effect was composed of the direct effect of Q_1 on C_2 at time 1 and the direct effect of C_2 at time 1 on C_2 at time 2.

Table 16 summarizes the model testing for the first path analysis. The equations shown in the table indicate the effects represented under each model. The first four models deal with effects on C_2 assessed at time 2. The equation for H_1 indicates that the natural logarithm of the odds of a passing as opposed to a failing response on the C_2 item at time 2 is a function only of a parameter representing the general mean of the log odds for the C_2 item. Thus, H_1 asserts no effects on C_2. The equation for H_2 represents the log odds for C_2 at time 2 as a function of the general mean and the main effect for Q_1. Thus, H_2 asserts that Q_1 affects C_2 at time 2. The equation for model H_3 asserts that C_2 at time 1 affects C_2 at time 2. Thus, H_3 represents generalization of C_2 at time 1 to C_2 at time 2. The equation for H_4 asserts that Q_1 affects both C_2 at time 1 and C_2 at time 2 and that C_2 at time 1 affects C_2 at time 2.

The four models in Table 16 are hierarchical. Models H_2 and H_3 are hierarchical with respect to H_1 and H_4 is hierarchical with respect to H_1, H_2, and H_3. Statistical comparisons for the models eventuated in the selection of H_4 as the preferred model. This model asserts that Q_1 affects C_2 at time 2 and C_2 at time 1 affects C_2 at time 2. Thus, model H_4 supports the occurrence of transfer from Q_1 to C_2 at time 2 and generalization from C_2 at time 1 to C_2 at time 2.

Although generalization did occur, it cannot be linked directly to the learning of Q_1. Model H_5 asserts independence between Q_1 and C_2 at time 1. This model indicates no effect of Q_1 on C_2 at time 1. Thus, Q_1 learning cannot be regarded as responsible for performance at time 1. The observed lack of generalization from the subordinate task Q_1 in the quadratic domain to the superordinate task C_2 in the cubic domain is consistent with the findings reported in Study 2.

The second path analysis examined generalization and transfer effects for tasks C_2 and C_3. Both task C_2 and task C_3 were assessed after C_2 training at time 2. In addition C_3 was assessed after C_3 training at time 3.

The second analysis examined transfer represented by the direct effect of

C_2 at time 2 on C_3 at time 3 and generalization represented by the indirect effect of C_2 at time 2 on C_3 at time 3. This indirect effect was reflected in the direct effect of C_2 at time 2 on C_3 at time 2 and the direct effect of C_3 at time 2 on C_3 at time 3.

Results of the second analysis are shown in Table 17. The five models tested are the same models examined in the first analysis. As in the first analysis, model H_4 was preferred. However, in contrast to the first analysis, model H_5 did not afford an acceptable fit for the data. This means that the hypothesis that at time 2 C_2 and C_3 were independent had to be rejected. The results support the assumption that C_2 generalized to C_3 at time 2. Accordingly, the preferred model indicated transfer from C_2 at time 2 to C_3 at time 3 and generalization in the form of an indirect effect of C_2 at time 2 on C_3 at time 3. This indirect effect involved generalization from C_2 at time 2 to C_3 at time 2 generalization from C_2 at time 2 to C_3 at time 3.

Discussion

The results of Study 3 support the hypothesis that prerequisitely ordered relations validated psychometrically will be congruent with prerequisitely ordered relations validated instructionally. The prerequisite ordering of Q_1 and C_2 and Q_1 and C_3 were maintained at each training phase of the study. Furthermore, the results afford no support for the differential forgetting hypothesis. Prerequisite ordering prevailed for the $Q_1 - C_2$ and $Q_1 - C_3$ task sets during retention as it did during the training phases of the study. However, from the response patterns for the retention data in Table 14 it seems that forgetting occurred, but mainly for transitional individuals. The trainees in transition for the $C_2 - C_3$ cross-classification on the test following

C_2 and C_3 instruction shifted to nonmastery classification on the retention test. Thus, forgetting occurred in the order in which learning occurred.

The results also provided support for the congruence of the positive effects hypothesis under psychometric and instructional validation conditions. There was a strong relationship between each of the quadratic and cubic domain items examined under the psychometric and the instructional approach.

The examination of generalization and transfer within and between domains provided information about how positive effects occur in hierarchical sequences. The present results indicate that positive effects between domains result from transfer. As Gagné (1977) suggested, mastery of a subordinate skill can facilitate the learning of a superordinate skill. By contrast, the results within domains indicate that positive effects can occur both as a result of generalization and transfer.

The fact that transfer taskes place between hierarchically ordered domains indicates that providing instruction in a subordinate skill can be beneficial. Nonetheless as pointed out in the discussion of study 2 there are advantages to initiating instruction at a superordinate level as opposed to a subordinate level. When instruction can be initiated at the superordinate level, downward generalization can occur. If trainees fail to profit from superordinate instruction, subordinate instruction may be provided with the expectation that transfer will occur.

In the case of within domain instruction, the present results also favor initiating instruction at the top of the domain. As the results of Study 2 show, downward generalization can occur within a domain. Nonetheless, if the individual, for any reason, cannot profit from instruction initiated at the higher levels in a domain, then instruction may be initiated at a lower level with the expectation that both generalization and transfer will occur.

Conclusion

The psychometric and instructional validation studies reported above have a number of implications regarding the advancement of knowledge about hierarchical sequences. The present findings provided empirical support for the idea of hierarchically ordered domain structures as opposed to the idea of hierarchically ordered tasks. The notion of domain structures raises new questions about generalization and transfer within hierarchical sequences. Whereas in the learning hierarchy model the major focus of research was on positive transfer between hierarchically ordered tasks, in the domain structure model the focus is on generalization and transfer both within and between domains.

The instructional validation findings reported here support the view that generalization and transfer may occur within domains. In addition they indicated that downward generalization and upward transfer may occur between domains. However, they failed to provide support for the assumption that upward generalization may occur between hierarchically ordered domains.

The findings regarding generalization and transfer within and between domains have important implications for training. Specifically, they suggest the desirability of targeting training ahead of the trainee's current level of functioning rather than at the current level of functioning. For many learners, training may profitably begin at the highest level in a domain rather than the lowest. Moreover, training may be initiated to advantage in a superordinate rather than a subordinate domain.

The findings on generalization and transfer suggest a learning process making use of something like the phenomena of top-down and bottom-up processing (see, for example, Anderson, 1980). By starting at a more advanced level than the level of current functioning, the individual is afforded the opportunity of conceptualizing simple problems as special cases of complex problems (top-down processing). The capability to accomplish this is apparent in the downward generalization that was observed to occur in the instructional validation stuides. By contrast, when instruction is initiated in a subordinate domain, top-down processing does not occur. The familiar problem of not being able to see the forest for the trees is revealed as evidenced by the absence of upward generalization.

As indicated earlier, there are undoubtedly limits with respect to how far ahead one can skip in a domain structure. As a consequence, progress through a hierarchical sequence may be thought of as combining bottom-up and top-down processing. In all likelihood the trainee uses top-down processing to acquire skills representing a subset of domains and then uses bottom-up processing to progress to a higher level subset of domains. As suggested previously, the determination of those factors affecting the degree to which skipping ahead is feasible represents a high research priority.

The present reserach has implications with respect to reserach strategies regarding hierarchical sequences as well as with regard to the advancement of knowledge about such sequences. As implied at the beginning of this article, although the notion of hierarchical sequences has been widely accepted for a long time, little is known about the hierarchical structure of knowledge. There is an obvious need for a rapid increase in information about the structure of knowledge in such areas as academic subject matter fields and technical specialties. The findings of the present research indicating the congruence of results for psychometric and instructional validation studies suggest the feasibility of employing the psychometric validation approach as a tool for gaining information about krowledge structures in a rapid and efficient manner.

One possible way to increase information about knowledge structures dramatically would be to link psychometric validation to test development. Bergan (in press) has suggested the need for a new kind of assessment that would reference examinee performance to position in a set of paths defining a knowledge structure. The present research suggests that path referencing would have the advantage of providing information that could be used to make predictions regarding generalization and transfer in hierarchical domain structures of the sort that might be used in instruction. Psychometric validation used for purposes of test development could provide the basis for hypotheses to be investigated using instructional validation techniques. The instructional studies in turn could provide information regarding the validity of domain structures determined psychometrically.

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APPENDIX A

Table 1

Sample Problems From Hypothesized Domains[1]

Domain	Problems
Term Transposition	$X/A = B$
	$X + A + B = C$
Distributive Property	$N(X+R) = Z$
	$\frac{A(X+B)}{C} = D$
Factoring	$NX + RX = Y$
	$(NX + RX)Y = Z$

[1] In each case the task was to solve for X.

Table 2

Cross-Classification of Two

Hypothesized Tasks

Task A	Task B: 1	2	3
1			
2			
3			

Table 3

Models Used in Establishing Item Domains[1]

H_o

		B		
		1	2	3
	1	E	E	I
A	2	E	E	I
	3	E	E	I

H_1

		B		
		1	2	3
	1	X	E	E
A	2	E	X	E
	3	E	E	X

H_2

		B		
		1	2	3
	1	X	I	I
A	2	E	E	I
	3	E	E	X

H_2'

		B		
		1	2	3
	1	X	I	I
A	2	I	I	I
	3	I	I	X

H_3

		B		
		1	2	3
	1	X	I	I
A	2	I	X	I
	3	E	E	X

H_3'

		B		
		1	2	3
	1	X	I	I
A	2	I	X	I
	3	I	I	X

H_4

		B		
		1	2	3
	1	X	I	I
A	2	X	I	I
	3	E	E	X

H_5

		B		
		1	2	3
	1	X	I	I
A	2	I	I	I
	3	X	X	X

1. The E's connected by curved lines indicate cells constrained to be equiprobable. The I's indicate cells for which the hypothesis of independence prevails. The X's indicate cells reflecting response patterns associated with specific latent classes.

Table 4

Observed Cross-Classifications for the Term-
Transposition Domain[1]

Response Patterns		Cross-Classifications					
Tasks							
A	B	A1 - M1	A1 - A2	A1 - M2	M1 - A2	M1 - M2	A2 - M2
1	1	65	72	69	82	99	97
1	2	4	2	2	12	2	22
1	3	6	1	4	9	2	16
2	1	14	22	28	14	16	6
2	2	12	12	4	13	10	10
2	3	12	4	6	8	9	10
3	1	24	19	38	17	20	10
3	2	19	40	20	29	14	22
3	3	161	145	146	133	145	124

1. The letters in the letter-number combinations labeling the columns below the cross-classifications heading indicate addition-subtraction (A) or multiplication-divison (M) problems. The numbers refer to the number of steps required for problem solution.

Table 5

Observed Cross-Classifications for the Factoring-Distributive Property Domain[1]

Response Patterns (Tasks) A	B	Cross-Classifications F2-F3	F2-F4	F2-F5	F2-D3	F2-D4	F2-D5	F3-F4	F3-F5	F3-D3	F3-D4	F3-D5	F4-F5	F4-D3	F4-D4	F4-D5	F5-D3	F5-D4	F5-D5	D3-D4	D3-D5	D4-D5
1	1	160	157	159	144	151	154	169	169	150	161	166	175	156	168	175	159	172	185	152	161	162
1	2	2	6	1	11	18	16	11	2	8	11	12	1	3	6	10	3	4	3	17	8	6
1	3	1	0	3	15	25	41	0	11	12	22	33	7	11	20	26	8	18	23	1	1	0
2	1	11	13	15	10	8	5	9	14	12	8	6	15	16	9	8	19	12	13	8	19	26
2	2	10	6	5	9	5	6	6	2	5	3	5	3	4	3	5	1	1	5	13	4	13
2	3	10	12	11	15	9	32	14	13	17	11	32	14	14	10	30	14	9	25	13	11	17
3	1	9	13	28	9	4	4	5	21	14	7	4	12	19	14	8	24	18	4	8	31	23
3	2	17	20	5	11	8	9	15	7	6	5	2	7	14	12	6	7	6	3	26	31	24
3	3	97	90	90	93	89	50	88	80	93	89	54	83	80	75	49	82	77	56	79	51	46

1. Letters in the cross-classification columns indicate factoring (F) and distributive property (D) problems. Numbers refer to the number of steps required for problem solution.

Table 6

Chi-squared Tests for the Term-transposition Domain[1]

Tasks	H_o		H_1		H_2		H_2'		H_3		H_3'		H_4		H_5	
A B	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p
A1-M1	200.65	<.01	19.04	<.01	1.18*	<.90	.29	<.90	1.19	<.50	.16	<.90	.87	<.75	.28	<.75
A1-A2	251.78	<.01	73.48	<.01	10.92	<.025	9.71	<.01	10.94	<.01	1.38*	<.25	7.73	<.025	.09	<.90
A1-M2	237.56	<.01	66.65	<.01	29.12	<.01	6.73	<.05	29.14	<.01	2.63*	<.25	5.75	<.10	.07	<.90
M1-A2	197.55	<.01	15.32	<.01	3.36*	<.05	1.93	<.50	3.37	<.25	1.35	<.25	3.27	<.25	.10	<.90
M1-M2	320.54	<.01	30.58	<.01	2.58*	<.50	.06	<.175	2.59	<.50	.04	<.90	1.07	<.75	.01	<.90
A2-M2	203.97	<.01	15.73	<.01	6.69	<.10	.41*	<.90	6.50	<.05	0	<.99	6.70	<.05	.19	<.75

[1]The letters in the letter-number combinations used in designating task pairs indicate the addition-subtractions (A) and multiplication-division(M) dimensions. Numbers refer to the number of steps required to achieve problem solution.

The degrees of freedom for H_o through H_5 are as follows:

H_o	5	H_3	2
H_1	3	H_3'	1
H_2	3	H_4	2
H_2'	2	H_5	1

Asterisks indicate preferred models.

Table 7

Chi-squared Tests for the Factoring-distributive Property Domain[1]

Tasks	H_o		H_1		H_2		H_2'		H_3		H_3'		H_4		H_5	
A B	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p
D3-F2	305.52	<.01	1.6086*	<.75	.33	<.975	.28	<.90	.34	<.90	.02	<.90	.32	<.90	.12	<.75
D3-F3	345.09	<.01	6.4418*	<.10	6.51	<.10	1.66	<.50	6.51	<.05	1.07	<.50	4.76	<.10	1.47	<.25
D3-F4	337.67	<.01	11.924	<.01	10.35	<.025	3.52	<.25	10.39	<.01	1.99	<.25	.67*	<.75	1.00	1.00
D3-F5	382.18	<.01	29.4	<.01	30.40	<.01	3.63	<.25	31.02	<.01	0	1	11.97	<.01	2.10	<.10
D3-D4	323.14	<.01	13.93	<.01	25.33	<.01	11.71	<.01	25.34	<.01	4.27*	<.05	25.34	<.01	11.28	<.01
D3-D5	285.98	<.01	49.99	<.01	15.28	<.01	13.55	<.01	15.35	<.01	2.25*	<.25	9.60	<.01	9.59	<.01
D4-F2	292.64	<.01	20.939	<.01	2.06*	<.75	2.02	<.50	2.07	<.50	1.18	<.50	1.53	<.25	.08	<.90
D4-F3	340.53	<.01	10.929	<.025	2.69*	<.50	.86	<.75	2.69	<.50	0	1	1.03	<.75	.69	<.50
D4-F4	320.66	<.01	1.85072*	<.75	4.54*	<.25	2.08	<.50	4.56	<.25	1.55	<.25	.15	<.95	4.00	1.00
D4-F5	363.86	<.01	4.79*	<.25	20.33	<.01	1.92	<.50	20.88	<.01	.67	<.50	6.66	<.05	.38	<.75
D4-D5	325.15	<.01	27.57	<.01	12.39	<.01	11.21	<.01	12.43	<.01	.86*	<.50	11.41	<.01	8.74	<.01
D5-F2	234.24	<.01	55.13	<.01	5.32*	<.25	2.10	<.50	5.32	<.10	.02	<.90	5.32	<.10	2.00	<.25
D5-F3	310.60	<.01	59.904	<.01	3.14*	<.50	3.14	<.25	3.14	<.25	1.83	<.25	3.15	<.25	2.20	<.25
D5-F4	305.29	<.01	12.4176	<.01	2.29*	<.75	2.06	<.50	2.29	<.50	.73	<.50	2.26	<.50	1.97	<.25
D5-F5	387.10	<.01	41.264	<.01	6.35*	<.10	1.12	<.75	6.35	<.05	1.12	<.50	.45	<.90	.30	<.75

1. The letters in the task descriptions refer to factoring (F) and distributive property (D) problems. The numbers indicate number of steps to problem solution.

* Asterisk indicates preferred model.

Table 7 (continued)

Tasks	H_o		H_1		H_2		H_2'		H_3		H_3'		H_4		H_5	
A B	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p	X^2	p
F2-F3	434.24	<.01	16.06	<.01	2.81*	<.50	1.66	<.50	2.82	<.25	0	< 1	2.80	<.25	.30	<.75
F2-F4	392.54	<.01	22.68	<.01	12.63	<.01	12.50	<.01	12.63	<.01	4.58*	<.05	11.85	<.01	10.36	<.01
F2-F5	420.47	<.01	40.26	<.01	23.67	<.01	.77	<.75	23.68	<.01	.06	<.90	17.24	<.01	.06	<.90
F3-F4	427.12	<.01	7.17*	<.01	23.69	<.01	20.09	<.01	23.70	<.01	5.92*	<.025	23.70	<.01	18.25	<.01
F3-F5	384.90	<.01	15.13	<.01	19.84	<.01	1.17	<.75	20.18	<.01	.52	<.50	7.38	<.025	.02	<.90
F4-F5	441.56	<.01	18.41	<.01	12.40	<.01	2.63	<.50	12.42	<.01	1.72	<.25	1.44*	<.50	.11	<.75

The degrees of freedom for the models are:

H_o	5
H_1	3
H_2	3
H_2'	2
H_3	2
H_3'	1
H_4	2
H_5	1

Table 8

Observed Cross-classifications for the Term-transposition and Factoring-distributive Property Domains

Response Patterns Tasks A	B	Cross-Classifications A_1-D_3	A_1-D_4	A_1-D_5	A_1-F_2	A_1-F_3	A_1-F_4	A_1-F_5
1	1	74	74	75	74	75	74	75
1	2	0	1	0	0	0	1	0
1	3	1	0	0	1	0	0	0
2	1	32	36	37	32	33	35	33
2	2	4	1	0	3	2	1	2
2	3	2	1	1	3	3	2	3
3	1	64	84	99	57	68	82	94
3	2	30	20	43	28	17	19	9
3	3	110	100	62	119	119	103	101

Tasks A	B	M_1-D_3	M_1-D_4	M_1-D_5	M_1-F_2	M_1-F_3	M_1-F_4	M_1-F_5
1	1	97	102	103	97	100	101	103
1	2	2	1	0	3	1	1	0
1	3	4	0	0	3	2	1	0
2	1	26	30	33	28	29	32	32
2	2	5	3	1	4	4	1	1
2	3	4	2	1	3	2	2	2
3	1	47	62	75	38	47	58	67
3	2	27	18	42	24	14	19	10
3	3	105	99	62	117	118	102	102

Table 8 (continued)

Response Patterns		Cross-classifications						
Tasks								
A	B	A_2-D_3	A_2-D_4	A_2-D_5	A_2-F_2	A_2-F_3	A_2-F_4	A_2-F_5
1	1	107	111	112	101	105	105	107
1	2	3	1	0	5	1	1	2
1	3	3	1	1	7	6	6	4
2	1	32	40	44	33	36	41	42
2	2	11	9	5	12	7	4	2
2	3	11	5	5	9	11	9	10
3	1	31	43	55	29	34	44	53
3	2	20	12	38	14	11	16	7
3	3	99	95	57	107	105	90	90

Tasks								
A	B	M_2-D_3	M_2-D_4	M_2-D_5	M_2-F_2	M_2-F_3	M_2-F_4	M_2-F_5
1	1	128	134	133	126	131	133	134
1	2	2	1	2	5	2	1	1
1	3	5	0	0	4	2	1	0
2	1	15	19	21	14	15	15	17
2	2	2	3	2	4	3	2	2
2	3	9	4	3	8	8	9	7
3	1	27	41	57	23	30	43	51
3	2	30	18	39	22	14	18	8
3	3	99	97	60	111	112	95	97

Table 9

Chi-squared Tests for the Cross-classification of the Term-transposition and Factoring-distributive Property Domains

	H_o			H_1			H_2			H_2'			H_3			H_3'			H_4			H_5		
	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p
A1-D3	245.08	5	<.01	153.54	3	<.01	41.61	3	<.01	9.74	2	<.01	41.64	2	<.01	3.90	1	<.05	14.50	2	<.01	1.92*	1	<.25
A1-D4	282.59	5	<.01	179.62	3	<.01	84.55	3	<.01	7.97	2	<.025	84.63	2	<.01	.21	1	<.75	43.40	2	<.01	1.03*	1	<.50
A1-D5	234.28	5	<.01	239.99	3	<.01	73.99	3	<.01	23.22	2	<.01	22.70	2	<.01	0.00	0	-	30.03	2	<.01	0.00*	0	-
A1-F2	255.42	5	<.01	137.93	3	<.01	41.75	3	<.01	11.49	2	<.01	42.54	2	<.01	3.40	1	<.10	11.34	2	<.01	.32*	1	<.75
A1-F3	294.32	5	<.01	150.83	3	<.01	65.96	3	<.01	4.45	2	<.25	68.62	2	<.01	0.00	0	-	34.69	2	<.01	0.00*	0	-
A1-F4	280.64	5	<.01	170.35	3	<.01	83.33	3	<.01	7.70	2	<.025	87.23	2	<.01	.42	1	<.75	44.08	2	<.01	1.73*	1	<.25
A1-F5	316.31	5	<.01	179.20	3	<.01	114.91	3	<.01	.35	2	<.90	117.96	2	<.01	0.00	0	-	84.37	2	<.01	0.00*	0	-
M2-D3	250.24	5	<.01	86.20	3	<.01	25.51	3	<.01	7.09	2	<.05	25.52	2	<.01	4.01	1	<.05	6.20	2	<.05	.72*	1	<.50
M1-D4	318.08	5	<.01	134.81	3	<.01	51.46	3	<.01	3.63	2	<.25	51.48	2	<.01	.39	1	<.75	26.51	2	<.01	.91*	1	<.50
M1-D5	266.43	5	<.01	199.83	3	<.01	47.55	3	<.01	18.63	2	<.01	49.95	2	<.01	0.00	1	1	13.85	2	<.01	0.00*	1	1
M1-F2	279.96	5	<.01	77.23	3	<.01	26.18	3	<.01	8.91	2	<.025	26.18	2	<.01	2.88*	1	<.10	3.25*	2	<.25	.07	1	<.90
M1-F3	326.77	5	<.01	94.16	3	<.01	44.45	3	<.01	3.51	2	<.25	44.45	2	<.01	2.39	1	<.25	19.75	2	<.01	.91*	1	<.50
M1-F4	319.03	5	<.01	124.34	3	<.01	59.12	3	<.01	9.77	2	<.01	60.16	2	<.01	1.02	1	<.50	22.60	2	<.01	.14*	1	<.75
M1-F5	368.08	5	<.01	143.06	3	<.01	84.07	3	<.01	3.09	1	<.25	86.57	2	<.01	0.00	0	-	49.61	2	<.01	0.00*	1	1
A2-D3	276.43	5	<.01	57.54	3	<.01	13.67	3	<.01	2.07*	2	<.50	13.67	2	<.01	.45	1	<.75	2.39*	2	<.25	0.00	1	1
A2-D4	338.43	5	<.01	101.86	3	<.01	40.79	3	<.01	.40	2	<.90	40.79	2	<.01	.27	1	<.75	18.69	2	<.01	.15*	1	<.75
A2-D5	266.18	5	<.01	155.94	3	<.01	42.17	3	<.01	18.70	2	<.01	7.93	2	<.05	3.73	1	<.10	4.45*	2	<.25	1.30	1	<.50

Table 9 (continued)

	H_o			H_1			H_2			H_2'			H_3			H_3'			H_4			H_5		
Tasks	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p	X^2	d.f.	p
A2-F2	264.70	5	<.01	38.62	3	<.01	18.94	3	<.01	1.44	2	<.50	18.96	2	<.01	1.42	1	<.25	6.08	2	<.05	.73*	1	<.50
A2-F3	308.47	5	<.01	63.73	3	<.01	40.61	3	<.01	3.29	2	<.25	40.61	2	<.01	3.11	1	<.10	13.87	2	<.01	1.54*	1	<.25
A2-F4	291.12	5	<.01	83.38	3	<.01	57.29	3	<.01	8.63	2	<025	57.70	2	<.01	5.14	1	<.025	17.03	2	<.01	.70*	1	<.50
A2-F5	328.79	5	<.01	95.31	3	<.01	87.38	3	<.01	1.82	2	<.50	87.75	2	<.01	.01	1	<.95	43.90	2	<.01	.62*	1	<.50
M2-D3	302.80	5	<.01	39.80	3	<.01	12.32	3	<.01	10.66	2	<.01	13.07	2	<.01	2.70*	1	<.25	3.36	2	<.25	.25	1	<.75
M2-D4	365.06	5	<.01	86.26	3	<.01	22.78	3	<.01	3.61	2	<.25	22.81	2	<.01	.75	1	<.50	10.74	2	<.01	1.53*	1	<.25
M2-D5	277.69	5	<.01	133.92	3	<.01	22.56	3	<.01	11.17	2	<.01	4.12*	2	<.25	.71	1	<.50	6.23	2	<.05	2.83	1	<.10
M2-F2	312.90	5	<.01	26.01	3	<.01	5.92*	3	<.25	4.11	2	<.25	5.93	2	<.01	.12	1	<.75	1.06	2	<.75	1.04	1	<.50
M2-F3	362.28	5	<.01	42.31	3	<.01	14.70	3	<.01	1.76	2	<.50	14.75	2	<.01	.01	1	<.95	6.61	2	<.05	.66*	1	<.50
M2-F4	344.26	5	<.01	69.21	3	<.01	21.81	3	<.01	2.71	2	<.50	10.62	2	<.01	.06	1	<.90	14.16	2	<.01	.84*	1	<.50
M2-F5	387.86	5	<.01	88.00	3	<.01	49.58	3	<.01	2.68	2	<.50	37.37	2	<.01	2.41	1	<.25	37.64	2	<025	2.68*	1	<.10

Table 10

Contingency Table for Study 1

Response Patterns* Tasks A	B	C	D	Q_2, Q_1	Q_6, Q_1	Q_7, Q_1	Q_8, Q_1	Q_6, Q_2	Q_7, Q_2	Q_8, Q_2	Q_7, Q_6	Q_8, Q_6	Q_8, Q_7
1	1	1	1	75	49	50	45	45	50	39	42	38	35
2	1	1	1	3	3	3	2	2	2	2	2	3	9
1	2	1	1	4	2	1	1	6	4	6	7	4	2
2	2	1	1	5	1	3	0	2	1	1	6	3	2
1	1	2	1	6	4	10	5	5	8	6	7	2	2
2	1	2	1	2	1	0	1	0	1	0	2	0	1
1	2	2	1	0	0	0	0	1	2	0	1	3	1
2	2	2	1	0	2	1	0	1	0	0	1	1	2
1	1	1	2	12	13	7	5	11	7	6	3	4	6
2	1	1	2	2	1	1	1	1	0	1	3	2	1
1	2	1	2	1	0	1	0	3	0	1	2	2	1
2	2	1	2	1	1	2	2	0	4	8	3	0	0
1	1	2	2	4	31	30	42	86	22	36	3	11	14
2	1	2	2	0	2	3	3	5	5	5	0	2	0
1	2	2	2	5	8	8	9	6	10	9	5	6	7
2	2	2	2	83	85	83	87	89	87	91	116	122	120

* 1 denotes correct response

2 denotes incorrect response

Table 11

Chi Square Tests for Study 1*

Cross Classifications	H_1			H_2			H_3			H_4			H_5		
	X^2_L	df	p	X^2_L	df	p	X^2_L	df	p	X^2_L	df	p	X^2_L	df	p
Q_1-Q_2	29.07	12	<.01	6.23	4	<.25	8.81*	6	.25	6.53	3	<.1	11.33	7	<.25
Q_1-C_2	180.84	12	<.01	17.58	4	<.01	63.83	6	<.01	2.49	3	<.5	5.62*	7	<.75
Q_1-C_1	159.48	12	<.01	12.68	4	<.01	54.67	6	<.01	3.22	3	<.5	7.13*	7	<.5
Q_1-C_3	306.49	12	<.01	36.37	4	<.01	107.81	6	<.01	4.20	3	<.25	8.11*	7	<.5
Q_2-C_2	121.1	12	<.01	13.88	4	<.01	48.69	6	<.01	5.56	3	<.25	7.08*	7	<.5
Q_2-C_1	96.36	12	<.01	7.7*	4	<.25	38.52	6	<.01	4.84	3	<.25	14.98	7	<.05
Q_2-C_3	219.93	12	<.01	22.56	4	<.01	79.31	6	<.01	2.79	3	<.5	5.72*	7	<.75
C_2-C_1	16.70*	12	<.05	4.01	4	<.5	4.17	6	<.75	4.49	3	<.25	12.51	7	<.1
C_2-C_3	33.6	12	<.01	8.57*	4	<.1	17.71	6	<.01	4.82	3	<.25	5.41	7	<.75
Q_1-C_3	63.31	12	<.01	8.70	4	<.1	19.39	6	<.01	1.05*	3	<.75	14.39	7	<.05

* Preferred model denoted by an asterick

Table 12

An Example of the Feedback for the 3 Trials*

<u>Trial 1</u>

$\vdots$	$X^2 + X - 6 = 0$	
	$\vdots$	$\vdots$
<u>Step 3</u>: Simplifying the equation	$(\overset{1}{X}\ \ \overset{2}{})(\overset{3}{X}\ \ \overset{4}{}) = 0$	Put in locations 1 and 3 factors of the 1st term (X^2) of the equation.
$\vdots$	$(\overset{1}{X}\ \ \overset{2}{3})(\overset{3}{X}\ \ \overset{4}{2}) = 0$	Put in locations 2 and 4 factors of the 3rd term (-6) of the equation.
	$(X + 3)(X - 2) = 0$	Between these locations go plus or minus signs. Remember, you need to get a combination of factors and signs that cross-multiply to equal the equation you're solving.
		$\vdots$
	$(\overset{1}{X} + \overset{2}{3})(\overset{3}{X} - \overset{4}{2}) = 0$	To cross multiply: Multiply the elements in locations 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$\vdots$	$\vdots$
	$X^2 + X - 6 = 0$	Collect terms and check to see if the result is the same as the equation.
	$\vdots$	$\vdots$

<u>Trial 2</u>

$\vdots$	$X^2 + 5X + 4 = 0$	
	$\vdots$	
<u>Step 3</u>: Simplifying the equation	$(\overset{1}{X} + \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$	Put in locastions 1 and 3 factors of the 1st term (X^2) of the equation at the end of step 2, and in location 2 and 4, put factors of the 3rd term (5). Between these put signs.
$\vdots$	$\vdots$	$\vdots$
	$(\overset{1}{X} + \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$	Now cross multiply to check and see if the two expressions are the ones you want. To cross multiply: Add together the four products obtained from multiplying the terms in 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$X^2 + 4X - 5 = 0$	Now collect terms and check to see if the result equals the equation at the end of step 2.
	$\vdots$	$\vdots$

<u>Trial 3</u>

$\vdots$	$X^2 + 3X - 4 = 0$	
	$\vdots$	
<u>Step 3</u>: Simplifying the equation.	$(X + 4)(X - 1) = 0$	
$\vdots$	$\vdots$	

*The dots ($\vdots$) signify that other steps, equations, and descriptions occurred before, in between, and after what has been speicifcally illustrated.

Table 13

Types of Rules and Strategies Supplied During Instruction

Step 1	Determining the Number of roots	$3X^2 + 9X - 12 = 0$	The highest exponent in the equation is a 2, therefore, there are two roots.
Step 2	Factoring and eliminating common elements	$3(X^2 + 3X - 4) = 0$ $X^2 + 3X - 4 = 0$	A three can be factored out and eliminated from the equation.
Step 3	Simplifying the equation	$(X + 4)(X - 1) = 0$	These are the two factored expe expressions which equal $X^2 + 3X - 4 = 0$
Step 4	Obtain roots	$X + 4 = 0,\ X = -4$ $X - 1 = 0,\ X = 1$	The roots are -4 and 1.
Step 5	Check solution	$3(-4)^2 + 9(-4) - 12 \stackrel{?}{=} 0$ $48 - 48 = 0$ $3(1)^2 + 9(1) - 12 \stackrel{?}{=} 0$ $12 - 12 = 0$	-4 and 1, when substituted into the equation, both set the equation equal to 0.

Table 14

Contingency Tables for Study 2

Response Patterns*					Cross Classifications for Training in Q_1			Cross Classifications for Training in Q_2			Cross Classifications for Training in Q_7			Cross Classifications for Training in Q_6		
Tasks				BB'												
A	A'	B	B'	AA'	Q_2,Q_1	Q_1,C_2	Q_1,C_1	Q_1,Q_2	Q_2,C_2	Q_2,C_1	Q_1,C_2	Q_2,C_2	C_2,C_1	Q_1,C_1	Q_2,C_1	C_2,C_1
1	1	1	1		12	1	1	20	1	2	11	13	8	15	10	10
2	1	1	1		0	0	0	2	0	0	2	0	1	1	2	0
1	2	1	1		0	0	0	0	0	0	2	1	0	1	5	0
2	2	1	1		0	0	0	1	0	0	4	5	10	3	3	0
1	1	2	1		2	0	0	4	0	0	1	0	0	0	0	0
2	1	2	1		0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	1		1	0	0	1	0	0	0	1	0	0	0	0
2	2	2	1		0	0	0	3	0	0	0	0	1	0	0	0
1	1	1	2		7	0	1	2	0	0	1	2	1	0	0	6
2	1	1	2		1	0	0	0	0	0	1	2	0	0	0	0
1	2	1	2		0	0	0	0	0	0	1	0	0	0	0	0
2	2	1	2		1	0	0	2	0	0	2	1	4	2	2	0
1	1	2	2		11	31	30	0	22	21	2	1	3	7	5	4
2	1	2	2		4	5	5	0	8	8	0	0	0	0	0	0
1	2	2	2		4	5	5	1	4	4	1	2	2	0	0	2
2	2	2	2		16	17	17	36	37	37	29	29	27	29	31	36

*1 denotes correct responses

2 denotes incorrect responses

Table 15

Chi Square Tests for Study 2*

Cross Classifications by Training		H1 X_L^2	H1 df	H1 p	H2 X_L^2	H2 df	H2 p	H3 X_L^2	H3 df	H3 p	H4 X_L^2	H4 df	H4 p	H5 X_L^2	H5 df	H5 p
Training in Q_1	Q_1-Q_2	63.38	12	<.01	7.12*	4	<.25	23.38	6	<.01	6.00	3	<.25	5.41	7	>.25
	Q_1-C_2	311.1	12	<.01	34.03	4	<.01	93.32	6	<.01	.27	3	<.95	.01*	7	>.5
	Q_1-C_1	295.6	12	<.01	31.02	4	<.01	87.14	6	<.01	.29	3	<.95	.007*	7	>.5
Training in Q_2	Q_1-Q_2	18.33*	12	<.25	7.39	4	<.25	11.37	6	<.1	-	3	-	-	7	-
	Q_2-C_2	209.4	12	<.01	15.72	4	<.01	53.96	6	<.01	.19	3	<.95	.01*	7	<.99
	Q_2-C_1	198.6	12	<.01	14.29	4	<.01	50.43	6	<.01	.19	3	<.95	.004*	7	<.99
Training in C_1	Q_1-C_1	13.61*	12	<.5	5.96	4	<.25	6.85	6	<.5	4.33	3	<.25	14.20	7	<.05
	C_2-C_1	62.00	12	<.01	9.50*	4	<.05	16.94	6	<.01	7.50	3	<.10	66.75	7	<.01
	Q_2-C_1	23.27*	12	<.01	15.10	4	<.01	16.87	6	<.01	15.05	3	<.01	25.39	7	<.01
Training in C_2	Q_1-C_2	50.00	12	<.01	14.93	4	<.05	15.21	5	<.05	.90*	3	<.95	21.06	7	<.01
	Q_2-C_2	34.56	12	<.01	10.01	4	<.05	8.57	6	<.25	.026*	3	<.99	27.63	7	<.01
	C_2-C_1	43.57	12	<.01	1.75*	4	<.90	10.82	6	<.1	.23	3	<.99	.03	7	<.99

* preferred model denoted by an asterick

Table 16

Contingency Tables for Study 3

Cross-Classifications

Response Patterns* Tasks A B C D	Training in Q_1 C_2-Q_1	C_3-Q_1	Training in Q_6 C_2-Q_1	C_3-Q_1	C_3-C_2	Training in Q_8 C_2-Q_1	C_3-Q_1	C_3-C_2	One week retention test C_2-Q_1	C_3-Q_1	C_3-C_2
1 1 1 1	14	11	74	51	51	75	55	48	65	52	51
2 1 1 1	2	1	0	0	2	0	0	6	3	3	3
1 2 1 1	0	0	4	3	1	0	1	1	0	0	1
2 2 1 1	0	0	0	0	0	1	1	2	0	0	0
1 1 2 1	5	3	2	8	8	17	19	14	6	9	6
2 1 2 1	0	0	0	0	0	0	0	4	0	0	1
1 2 2 1	0	0	0	1	0	1	0	1	1	1	3
2 2 2 1	0	0	0	0	1	0	1	1	0	1	1
1 1 1 2	0	5	18	7	5	5	16	8	7	2	0
2 1 1 2	0	0	0	0	0	0	0	4	0	0	0
1 2 1 2	0	0	5	1	3	0	1	1	0	0	1
2 2 1 2	0	0	4	0	0	1	0	4	0	0	1
1 1 2 2	104	104	26	54	14	37	43	6	51	66	11
2 1 2 2	4	5	2	2	0	0	0	4	3	3	3
1 2 2 2	7	7	10	14	23	4	3	3	2	2	2
2 2 2 2	36	36	27	31	64	31	31	65	29	28	83

*: 1 denotes correct response

2 denotes incorrect response

Table 17

Chi Square Tests for
Study 3 *

Cross Classification by Training		H_1			H_2			H_3			H_4			H_5		
		X^2_L	df	p	X^2_L	df	p	X^2_L	df	p	X^2_L	df	p	X^2_L	df	p
Training in Q_1	Q_1-C_2	1131	12	<.01	165.70	4	<.01	396.80	6	<.01	1.62	3	<.75	5.69*	7	<.75
	Q -C_3	1099	12	<.01	152.30	4	<.01	387.60	6	<.01	1.33	3	<.75	1.89*	7	<.95
Training in C_2	Q_1-C_2	154.20	12	<.01	42.94	4	<.01	74.19	6	<.01	29.47	3	<.01	27.38*	7	<.01
	Q_1-C_3	414.60	12	<.01	48.17	4	<.01	136.7	6	<.01	2.71	3	<.5	6.69*	7	<.75
	C_2-C_3	127.70	12	<.01	18.14	4*	<.01	55.20	6	<.01	16.63	3	<.01	14.37	7	<.01
Training in C_3	Q_1-C_2	293.50	12	<.01	30.28	4	<.01	108.15	6	<.01	6.75	3	<.05	6.69*	7	<.5
	Q_1-C_3	325.60	12	<.01	21.81	4	<.01	108.97	6	<.01	5.64	3	<.25	6.02*	7	<.5
	C_2-C_3	23.24	12	<.05	7.50*	4	>.1	14.29	6	<.05	8.14	3	<.05	11.26	7	<.10
One Week Retention Test	Q_1-C_2	453.50	12	<.01	65.82	4	<.01	161.68	6	<.01	10.77	3	<.05	7.71*	7	<.5
	Q_1-C_3	635.00	12	<.01	91.66	4	<.01	240.70	6	<.01	9.71	3	<.05	11.37*	7	<.10
	C_2-C_3	47.68	12	<.01	13.70	4	<.01	27.75	6	<.01	7.98	3	<.05	9.34*	7	<.25

* Denotes preferred model

Table 18

Positive Effects Models for Q_1 and C_2

Model	Equations	X^2_L	df	p
H_1	$\Phi^C_{ij} = \beta^{\bar{C}}$	43.63	3	0.000
H_2	$\Phi^C_{ij} = \beta^{\bar{C}} + \beta^{A\bar{C}}_i$	15.36	2	0.000
H_3	$\Phi^C_{ij} = \beta^{\bar{C}} + \beta^{B\bar{C}}_j$	26.80	2	0.000
H_4	$\Phi^C_{ij} = \beta^{\bar{C}} + \beta^{A\bar{C}}_i + \beta^{B\bar{C}}_j$	0.00*	0	1.00
H_5	$\Phi^B_i = \beta^{\bar{B}}$	1.55	1	$<.25$

In the equations, A represents Q_1 assessed at time 1, B represents C_2 assessed at time 1, and C represents C_2 measured at time 2. Φ^C_{ij} indicates the natural logarithm that variable C will be passed as opposed to failed when variables A and B are at levels i and j respectively ($i = 0, 1$; $j = 0, 1$). The term $\beta^{\bar{C}}$ refers to the mean of the natural logarithm of the odds that variable C will be passed as opposed to failed for all values of variables A and B. The $\beta^{A\bar{C}}_i$ indicates the main effect of A on the log odds that C will be passed rather than failed when A is at level i ($i = 0, 1$). The other terms are similarly defined. The asterick denotes the preferred model.

Table 19

Positive Effects Models for C_2 and C_3

Model	Equations	X^2_L	df	p
H_1	$\Phi^{C}_{ij} = \beta^{\bar{C}}$	49.93	3	0.000
H_2	$\Phi^{C}_{ij} = \beta^{\bar{C}} + \beta^{A\bar{C}}_{i}$	7.26	2	0.026
H_3	$\Phi^{C}_{ij} = \beta^{\bar{C}} + \beta^{B\bar{C}}_{j}$	26.80	2	0.000
H_4	$\Phi^{C}_{ij} = \beta^{\bar{C}} + \beta^{A\bar{C}}_{i} + \beta^{B\bar{C}}_{j}$	.41*	1	< .75
H_5	$\Phi^{B}_{i} = \beta^{\bar{B}}$	63.47	1	0.000

In the equations, A represents C_2 assessed at time 2, B represents C_3 assessed at time 2, and C represents C_3 measured at time 3. The equations are defined as they were in Table 7. The asterick notes the preferred model.

APPENDIX B

Q_1 Instructions for Studies 2 and 3

The next phase of this instructional program will present the steps necessary to solve for quadratic equations. Equations of this kind involve more than one value for the unknown X. Your goal is to simplify the given equation into an equation with separate expressions each containing an X. Each of these expressions is used to attain a value for X.

STEP A Determining the number of values for X	$X^2 + 6X + 5 = 0$	The highest exponent in the initial equation refers to the number of X values we are solving for.
	$X^2 + 6X + 5 = 0$	There are 2 values for X since the highest exponent in the initial equation is 2.
STEP B	$(\quad\quad)(\quad\quad) = 0$	Set up your sets of parantheses.
Simplify	$(\overset{1}{\underline{X}}\ \overset{2}{\underline{\ }})(\overset{3}{\underline{X}}\ \overset{4}{\underline{\ }}) = 0$	Put in locations 1 and 3 factors of the 1st term (X^2) in the equation.
	$(\overset{1}{\underline{X}}\ \overset{2}{\underline{1}})(\overset{3}{\underline{X}}\ \overset{4}{\underline{5}}) = 0$	Put in locations 2 and 4 factors of the 3rd term (5) in the equation.
	$(\overset{1}{X} \oplus \overset{2}{1})(\overset{3}{X} \oplus \overset{4}{5}) = 0$	Between these locations go plus or minus sings. <u>Remember</u>, you need to get a combination of factors and signs that cross multiply to equal the equation you are solving.
	$(\overset{1}{X} + \overset{2}{1})(\overset{3}{X} + \overset{4}{5}) = 0$	Check to see if these two expressions cross multiply to equal the equation you are solving.
	$(\overset{1}{X} + \overset{2}{1})(\overset{3}{X} + \overset{4}{5}) = 0$ X^2, 5X 1X, 5	To cross multiply: multiply the elements in location 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$X^2 + 5X + 1X + 5 = 0$	Add these four multiplicative products together.
	$X^2 + 6X + 5 = 0$	Collect terms and check to see if the result is the same as the step A equati If the result does not equal the step A equation, you must try other combinations of factors or signs in the parentheses (return to Step B).

STEP C

Solve for the X values

$X + 1 = 0$ and

$X + 5 = 0$

If the result equals the step A equation, solve for the X values by setting each expression equal to 0.

$(X + 1) - 1 = 0 - 1$

$X = -1$

$(X + 5) - 5 = 0 - 5$

$X = -5$

The X values are then obtained by solving each of these equations. Thus, the two values that will make the original equatio equal 0 have been found.

Thus the two values for X in the equation are:

$X = -1$

$X = -5$

STEP D(optional)

Check your solution

$(-1)^2 + 6(-1) + 5 = 0$

$1 + (-6) + 5 = 0$

$(-5)^2 + 6(-5) + 5 = 0$

$25 + (-30) + 5 = 0$

You can check to see if these are the correct values by inserting each of them into the original equation and checking whether or not the solution to the equation is 0.

page 3

Now you try to solve the equation:

$X^2 + 5X + 4 = 0$

Step A Determine the number of values for X	$X^{2} + 5X + 4 = 0$	There are 2 values of X to solve for, since 2 is the highest exponent.
Step B Simplify the equation	$\overset{1}{(\_}\ \overset{2}{\_)}\overset{3}{(\_}\ \overset{4}{\_)} = 0$	Set up your parentheses.
	$(\overset{1}{\underline{X}} + \overset{2}{\underline{4}})\ (\overset{3}{\underline{X}} + \overset{4}{\underline{1}}) = 0$	Put in locations 1 and 3 factors of the 1st term (X^2) in the equation, and in locations 2 and 4 factors of the 3rd term (4) of the equation. Finally, put some signs between these locations. Remember, you need to get a combination of factors and signs that crossmultiply to equal the equation you're solving.
	$(\overset{1}{X} + \overset{2}{4})\ (\overset{3}{X} + \overset{4}{1}) = 0$ $X^2 + X + 4X + 4 = 0$	Crossmultiply to check and see if the two expressions are the ones you want. Crossmultiply and add together the four products obtained from multiplying the elements in location 1 and 2, 1 and 4, 2 and 3, and 2 and 4.
	$X^2 + 5X + 4 = 0$	Collect terms and check to see if the result equals the Step A equation. If the result does not equal the equation in Step A, try other combinations of factors or signs in the parentheses (return to the beginning of Step B).
Step C Solve for X values	$X + 4 = 0$ $(X+4) - 4 = 0-4$ $\underline{X = -4}$ $X + 1 = 0$ $(X+1) - 1 = 0-1$ $\underline{X = -1}$	If the result equals the Step A equation, solve for the X values by setting each expression equal to 0, and then solving each equation for X. Thus, two values that will make the original equation equal to 0 have been found. The 2 values for X are: $X = -4$ $X = -1$

 Q_1

Now, you try to solve the following equation:

$X^2 + 7X + 12 = 0$

Here is the solution:

$$X^2 + 7X + 12 = 0$$

$$(\overset{1}{X} + \overset{2}{4})(\overset{3}{X} + \overset{4}{3}) = 0$$

$$X^2 + 3X + 4X + 12 = 0$$

$$X + 4 = 0$$

$$\underline{X = -4}$$

$$X + 3 = 0$$

$$X = -3$$

Q_2 Instructions for Study 2

The next phase of this instructional program will present the steps necessary to solve quadratic equations. Equations of this kind involve more than one value for the unknown X. Your goal is to simplify the given equation with separate expressions, each containing an X. Each of these expressions is used to attain a value for X.

<u>Step A</u>	$3X^2 + 3X - 18 = 0$	The highest exponent in the initial equation refers to the number of X values we are solving for.
Determine the number of values for X	$3X^{2} + 3X - 18 = 0$	There are 2 values for X since the highest exponent in the initial equation is 2.
<u>Step B</u>	$3(X^2 + X - 6) = 0$	Since there is a highest factor greater than 1 common to all the terms of the equation, this number can be factored out of the equation.
Factor out common elements	$\frac{3(X^2 + X - 6)}{3} = \frac{0}{3}$	Dividing both sides of the equation by this factor leaves you with an equation that is simpler to solve.
	$X^2 + X - 6 = 0$	If you solve this simpler equation you will also have the values that solve the initial equation.
<u>Step C</u>	$(\quad)(\quad) = 0$	Set up your sets of parentheses.
Simplify the equation	$(\overset{1}{\underline{X}}\ \overset{2}{\underline{\ \ }})(\overset{3}{\underline{X}}\ \overset{4}{\underline{\ \ }}) = 0$	Put in locations 1 and 3 factors of the 1st term (X^2) of the simpler equation.
	$(\overset{1}{\underline{X}}\ \overset{2}{\underline{3}})(\overset{3}{\underline{X}}\ \overset{4}{\underline{2}}) = 0$	Put in locations 2 and 4 factors of the 3rd term (-6) of the simpler equation.
	$(\overset{1}{\underline{X}} + \overset{2}{\underline{3}})(\overset{3}{\underline{X}} - \overset{4}{\underline{2}}) = 0$	Between these locations go plus or minus signs. <u>Remember</u>, you need to get a combination of factors and signs that cross multiply to equal the equation you're solving.
	$(\overset{1}{X} + \overset{2}{3})(\overset{3}{X} - \overset{4}{2}) = 0$	Check to see if these two expressions cross multiply to equal the equation you're solving.

$$\overset{1}{(X} + \overset{2}{3)}\overset{3}{(X} - \overset{4}{2)} = 0$$

To cross multiply: Multiply the elements in location 1 and 3, 1 and 4, 2 and 3, and 2 and 4.

X^2, $-2X$, $3X$, -6

$$X^2 + (-2X) + 3X + (-6) = 0$$

Add these four multiplicative products together.

$$X^2 + X - 6 = 0$$

Collect terms and check to see if the result is the same as the simpler equation.

If the result does not equal the equation at the end of Step B, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

Step D

Solving for X

$X + 3 = 0$
and
$X - 2 = 0$

If the result does equal the equation at the end of Step B, solve for the X values by setting each expression equal to 0.

$X + 3 = 0$	$X - 2 = 0$
$(X+3)\ -3 = 0 - 3$	$(X-2\_+2 = 0$
$X = -3$	$X = 2$

The X values are then obtained by solving each of these equations. Thus, the two values that will make the original equation equal 0 have been found.

Step E

(Optional)

$3(-3)^2+3(-3)-18 = 0$

$3(9)+(-9)-18 = 0$

$27 - 27 = 0$

$3(2)^2+3(2)-18 = 0$

$3(4)+6-18 = 0$

$12 + 6 - 18 = 0$

Now insert each of these values into the original equation and check to see whether the answer to the equation equals 0.

Now you try to solve the equation:

$$2X^2 + 8X - 10 = 0$$

page 4 Q_2

Step A

Determining the number of values of X

$2X^{2} + 8X - 10 = 0$

There are 2 values of X to solve for, since 2 is the highest exponent.

Step B

Factor out common elements

$2(X^2 + 4X - 5) = 0$

Factor out any highest number common to all the terms in the equation.

$\frac{2(X^2 + 4X - 5)}{2} = \frac{0}{2}$

Dividing both sides of the equation by this factor leaves an equation that is easier to solve.

$X^2 + 4X - 5 = 0$

Step C

Simplify the equation

$(\quad)(\quad) = 0$

Set up your parentheses.

$(\overset{1}{X} + \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$

Put in locations 1 and 3 factors of the first term (X^2) of the equation at the end of step B, and in locations 2 and 4 put factors of the 3rd term (5). Between these locations put signs. Remember, you need to find combinations of factors and signs that cross multiply to equal the equation you're solving.

$(\overset{1}{X} + \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$

$X^2 - X + 5X - 5 = 0$

Now cross multiply to check and see if the two expressions are the ones you want. To cross multiply: Add together the four products obtained from multiplying the terms in locations 1 and 3, 1 and 4, 2 and 3, 2 and 4.

$X^2 + 4X - 5 = 0$

Now collect terms and check to see if the result equals the equation at the end of step B.

If the result does not equal the equation you must try other combinations of factors and signs (return to Step C).

Step D

Solve for the X values

$X + 5 = 0$
$(X + 5) - 5 = 0 - 5$
$X = -5$

$X - 1 = 0$
$(X - 1) + 1 = 0 + 1$
$X = +1$

If the result does check, we solve for the X values by setting each expression equal to 0, and then solving each equation for X. The result will be the 2 values that will make the original equation equal 0. You can check these values by inserting them into th original equation and checki to ee whether your answer is 0.

Now you try to solve the following equation:

$$3X^2 + 9X - 12 = 0$$

Here is tne solution:

$$3X^2 + 9X - 12 = 0$$

$$3(X^2 + 3X - 4) = 0$$

$$X^2 + 3X - 4 = 0$$

$$(\overset{1}{X} + \overset{2}{4})(\overset{3}{X} - \overset{4}{1}) = 0$$

$X = 4 = 0$	$X - 1 = 0$
$X = -4$	$X = 1$

The next phase of this instructional program will present the steps necessary to solve cubic equations. Equations of this kind involve more than one value for the unknown X. Your goal is to simplify the given equation into an equation with separate expressions each containing an X. Each of these expressions is used to attain a value for X.

<u>Step A</u>		
Determine the number of values for X	$4X^3 - 20X^2 + 24X = 0$	The highest exponent in the initial equation refers to the number of X values we are solving for.
	$4X^{3} - 20X^2 + 24X = 0$	There are 3 values for X since the highest exponent in this equation is 3.
<u>Step B</u> Factor out common elements	$4X(X^2 - 5X + 6) = 0$	Since the variable X is common to all the terms of the equation, factor it out of the equation.
<u>Step C</u>	$4X(\quad)(\quad) = 0$	Set up your sets of parentheses.
Simplify the equation	$4X(\overset{1}{\underline{X}}\quad\overset{2}{\underline{\ \ }})(\overset{3}{\underline{X}}\quad\overset{4}{\underline{\ \ }}) = 0$	Put in locations 1 and 3 factors of the 1st term (X^2) of the equation within the parentheses in step B.
	$4X(\overset{1}{\underline{X}}\quad\overset{2}{\underline{3}})(\overset{3}{\underline{X}}\quad\overset{4}{\underline{2}}) = 0$	Put in locations 2 and 4 factors of the 3rd term (6) of the equation within the parentheses in step B.
	$4X(\overset{1}{\underline{X}} - \overset{2}{\underline{3}})(\overset{3}{\underline{X}} - \overset{4}{\underline{2}}) = 0$	Between the locations go minus or plus signs. Remember, you need to get a combination of factors and signs that will cross multiply to equal the equation within the parenthesis in step B.
	$4X(\underline{X} - \underline{3})(\underline{X} - \underline{2}) = 0$ $X^2, -2X, -3X, 6$	To cross multiply: multiply together the elements in location 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$4X(X^2 + (-2x) + (-3X) + 6 = 0$	Add these four multiplicative products together.
	$4X(X^2 - 5X + 6) = 0$	Collect these terms and check to see if the result is the same as the equation within the parentheses in step B.
		If the result does not equal the step B equation, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

Step D

Solve for X values

$4X = 0$

$X - 3 = 0$

$X - 2 = 0$

If the result equals the step B equation, solve for the X values by setting each of the expressions in step C equal to 0.

$4X = 0$

$X = 0$

$X - 3 = 0$

$(X - 3) + 3 = 0 + 3$

$X = 3$

$X - 2 = 0$

$(X - 2) + 2 = 0 + 2$

$X = 2$

Solve each of these equations to get the three X values.

Step E (Optional)

Check solution

$4(0)^3 - 20(0)^2 + 24(0) = 0$

$0 - 0 + 0 = 0$

$4(3)^3 - 20(3)^2 + 24(3) = 0$

$4(27) - 20(9) + 72 = 0$

$180 - 180 + 72 = 0$

$180 - 180 = 0$

$4(2)^3 - 20(2)^2 + 24(2) = 0$

$4(8) - 20(4) + 48 = 0$

$32 - 80 + 48 = 0$

$80 - 80 = 0$

To check if these are the correct values, insert each of them into the original equation and check to see if your answer is 0.

Now you try to solve for X in the equation:

$3X^3 - 18\ X^2 + 15X = 0$

Step A Determine the number of values for X	$3X^{3} - 18X^2 + 15X = 0$	There are 3 values to solve for, since 3 is the highest exponent.
Step B Factor out common elements	$3X(X^2 - 6X + 5) = 0$	Since the variable X is common to all the terms of the equation, it can be factored out.
Step C Simplify the equation	$3X(\quad)(\quad) = 0$	Set up your parentheses.
	$3X(\overset{1}{X} - \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$	Put in locations 1 and 3 factors of the 1st term of the equation within the parentheses in step B(X^2), and in location 2 and 4 factors of the 3rd term of the equation within the parentheses in step B(5). Finally, put some signs between these locations. Remember, you need to get a combination of factors and signs that cross multiply to equal the equation you're solving.
	$3X(\overset{1}{X} - \overset{2}{5})(\overset{3}{X} - \overset{4}{1}) = 0$ $3X(X^2 + (-X)+(-5X)+5 = 0$	Cross multiply to check and see if the two expressions are the ones you want. Cross multiply and add together the four products obtained from multiplying the elements in location 1 and 2, 1 and 4, 2 and 3, 2 and 4.
	$3X(X^2 - 6X + 5) = 0$	Collect terms and check to see if the result equals the equation in step B. If the result does not equal the equation in step B, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).
Step D Solve for X values	1 $\begin{cases} 3X = 0 \\ X = 0 \end{cases}$ 2 $\begin{cases} X - 5 = 0 \\ (X - 5) + 5 = 0 + 5 \\ X = 5 \end{cases}$ 3 $\begin{cases} X - 1 = 0 \\ (X - 1) + 1 = 0 + 1 \\ X = 1 \end{cases}$	If the result equals the step B equation solve for the X values by setting each of the expressions in step C equal to 0. Then solving for X in each of these equations gives us the 3 values for X that solve the initial equation.

Now you try to solve for X in the equation:

$$2X^3 - 8X^2 + 6X = 0$$

Here is the solution:

$$2X^3 - 8X^2 + 6X = 0$$

$$2X(X^2 - 4X + 3) = 0$$

$$2X(\overset{1}{X} - \overset{2}{3})(\overset{3}{X} - \overset{4}{1}) = 0$$

$$2X(X^2 - X - 3X + 3) = 0$$

$$2X = 0$$

$$X = 0$$

$$X - 3 = 0$$

$$X = 3$$

$$X - 1 = 0$$

$$X = 1$$

C_2 Instructions for Studies 2 and 3

The next phase of this instructional program will present the steps necessary to solve cubic equations. Equations of this kind involve more than one value for the unknown X. Your goal is to simplify the given equation into an equation with separate expressions each containing an X. Each of these expressions is used to attain a value for X.

<u>STEP A</u> Determine the number of values for X	$3X^3 + 7X^2 + 2X = 0$	The highest exponent in the initial equation refers to the number of X values we are solving for.
	$3X^{3} + 7X^2 + 2X = 0$	There are 3 values for X since the highes exponent in this equation is 3.
<u>STEP B</u> Factor out common elements	$X(3X^2 + 7X + 2) = 0$	Since the variable X is common to all the terms of the equation, factor it out of the equation.
	$X(\quad)(\quad) = 0$	Set up your sets of parentheses.
<u>STEP C</u> Simplify the equation	$X(\overset{1}{\underline{3X}}\ \ \overset{2}{\underline{\ \ }})(\overset{3}{\underline{X}}\ \ \overset{4}{\underline{\ \ }}) = 0$	Put in locations 1 and 3 factors of the 1st term ($3X^2$) of the equation within the parentheses in step B.
	$X(\overset{1}{\underline{3X}}\ \ \overset{2}{\underline{1}})(\overset{3}{\underline{X}}\ \ \overset{4}{\underline{2}}) = 0$	Put in locations 2 and 4 factors of the 3rd term (2) of the equation within the parentheses in step B.
	$X(\overset{1}{\underline{3X}} + \overset{2}{\underline{1}})(\overset{3}{\underline{X}} + \overset{4}{\underline{2}}) = 0$	Between the locations go minus or plus signs. <u>Remember</u>, you need to get a combination of factors and signs that will cross multiply to equal the equation within the parentheses in step B.
	$X(\overset{1}{3X} + \overset{2}{1})(\overset{3}{X} + \overset{4}{2}) = 0$ $3X^2, 6X, 1X, 2$	To cross multiply: multiply together the elements in location 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$X(3X^2 + 6X + 1X + 2) = 0$	Add these four multiplicative products together.
	$X(3X^2 + 7X + 2) = 0$	Collect these terms and check to see if the result is the same as the equation within the parentheses in step B.
		If the result does not equal the step B equation, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

STEP D

Solve for X values

$X = 0$

$3X + 1 = 0$

$X + 2 = 0$

If the result equals the step B equation, solve for the X values by setting each of the expressions in step C equal to 0.

$\underline{X = 0}$

$3X + 1 = 0$

$(3X + 1) - 1 = 0 - 1$

$3X = -1$

$1/3(3X) = (-1)1/3$

$\underline{X = -1/3}$

Solve each of these equations to get the three X values.

$X + 2 = 0$

$(X + 2) - 2 = 0 - 2$

$\underline{X = -2}$

Thus, the three values for X in the equati are:

$X = 0$

$X = -1/3$

$X = -2$

STEP E (optional)

Check solution

$3(0^3) + 7(0)^2 + 2(0) = 0$

$0 + 0 + 0 = 0$

$3(-1/3)^3 + 7(-1/3)^2 + 2\ (-1/3) = 0$

$3\ (-1/27) + 7(1/9) + (-2/3) = 0$

To check if these are the correct values, insert each of the three values for X into the original equation and check to see if your answer is 0.

$3(2)^3 + 7(-2)^2 + 2(-2) = 0$

$3(-8) + 7(-2)^2 + 2(-2) = 0$

Now you try to solve for X in the equation:

$$2X^3 + 9X^2 + 4X = 0$$

STEP A

Determine the number of values for X

$$2X^{3} + 9X^2 + 4X = 0$$

There are 3 values to solve for, since 3 is the highest exponent.

STEP B

Factor out common elements

$$X(2X^2 + 9X + 4) = 0$$

Since the variable X is common to all the terms of the equation, it can be factored out.

STEP C

Simplify the equation

$$X(\_\ \ \_)(\_\ \ \_) = 0$$

Set up your parentheses.

$$X(\overset{1}{2X} + \overset{2}{1})(\overset{3}{X} + \overset{4}{4}) = 0$$

Put in locations 1 and 3 factors of the 1st term of the equation within the parentheses in step B ($2X^2$), and in location 2 and 4 factors of the 3rd term of the equation within the parentheses in step B (4). Finally, put some signs between these locations. Remember, you need to get a combination of factors and signs that cross multiply to equal the equation you're solving.

$$X(\overset{1}{2X} + \overset{2}{1})(\overset{3}{X} + \overset{4}{4}) = 0$$

$$X(2X^2 + 8X + 1X + 4) = 0$$

Cross multiply to check and see if the two expressions are the ones you want. Cross multiply and add together the four products obtained from multiplying the elements in location 1 and 2, 1 and 4, 2 and 3, and 2 and 4.

$$X(2X^2 + 9X + 4) = 0$$

Collect terms and check to see if the result equals the equation in step B.

If the result does not equal the equation in step B, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

STEP D

Solve for X values

1 $$X = 0$$

2
$$2X + 1 = 0$$
$$(2X + 1) - 1 = 0 - 1$$
$$\tfrac{1}{2}(2X) = (-1)\tfrac{1}{2}$$
$$X = -\tfrac{1}{2}$$

If the result equals the step B equation, solve for the X values by setting each of the expressions in step C equal to 0. Then solving for X in each of these equations gives us the 3 values for X that solve the initial equation.

3
$$X + 4 = 0$$
$$(X+4) - 4 = 0-4$$
$$X = -4$$

The 3 values for X are:

$X = 0$

$X = -\tfrac{1}{2}$

$X = -4$

Now you try to solve for X in the equation:

$$3X^3 + 10X^2 + 3X = 0$$

Here is the solution:

$3X^3 + 10X^2 + 3X = 0$

$X(3X^2 + 10X + 3) = 0$

$X(3X + 1)(X + 3) = 0$

$X(3X^2 + 9X + 1X + 3) = 0$

$\underline{X = 0}$

$3X + 1 = 0$

$\underline{X = -1/3}$

$X + 3 = 0$

$X = -3$

The next phase of this instructional program will present the steps necessary to solve cubic equations. Equations of this kind involve more than one value for the unknown X. Your goal is to simplify the given equation into an equation with separate expressions each containing an X. Each of these expressions is used to attain a value for X.

<u>STEP A</u> Determine the number of values for X	$24X^3 + 28X^2 + 8X = 0$	The highest exponent in the initial equation refers to the number of X values we are solving for.
	$24X^{3} + 28X^2 + 8X = 0$	There are 3 values for X since the highest exponent in this equation is 3.
<u>STEP B</u> Factor out common elements	$X(24X^2 + 28X + 8) = 0$	Since the variable X is common to all the terms of the equation, factor it out of the equation.
	$4X(6X^2 + 7X + 2) = 0$	Since there is a highest common factor greater than 1 common to all the terms of the equation, this number can also be factored out of the equation. In this case that number is 4.
<u>STEP C</u> Simplify the equation	$4X(\quad)(\quad) = 0$	Set up your sets of parentheses.
	$4X(\overset{1}{3X}\ \overset{2}{\ }\ )(\overset{3}{2X}\ \overset{4}{\ }\ ) = 0$	Put in locations 1 and 3 factors of the 1st term $(6X^2)$ of the equation within the parentheses at the end of Step B.
	$4X(\overset{1}{3X}\ \ \overset{2}{2})(\overset{3}{2X}\ \ \overset{4}{1}) = 0$	Put in locations 2 and 4 factors of the 3rd term (2) of the equation within the parentheses at the end of Step B.
	$4X(\overset{1}{3X} + \overset{2}{2})(\overset{3}{2X} + \overset{4}{1}) = 0$	Between the locations go minus or plus signs. Remember, you need to get a combination of factors and signs that will cross multiply to equal the equation within the parentheses at the end of Step B
	$4X(3X + 2)(2X + 1) = 0$	To cross multiply: multiply together the elements in location 1 and 3, 1 and 4, 2 and 3, and 2 and 4.
	$6X^2$, $3X$, $4X$, 2	
	$4X(6X^2 + 3X + 4X + 2) = 0$	Add these four multiplicative products together.
	$4X(6X^2 + 7X + 2) = 0$	Collect these terms and check to see if the result is the same as the equation within the parentheses in Step B.

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If the result does not equal the Step B equation, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

STEP D
Solve for X values

$4X = 0$

$3X + 2 = 0$

$2X + 1 = 0$

If the result equals the Step B equation, solve for the X values by setting each of the expressions in Step C equal to 0.

$4X = 0$

$\underline{X = 0}$

Solve each of these equations to get the three X values.

$3X + 2 = 0$

$(3X + 2) - 2 = 0-2$

$3X = -2$

$1/3(3X) = (-2)1/3$

$\underline{X = -2/3}$

$2X + 1 = 0$

$(2X + 1) - 1 = 0-1$

$2X = -1$

$1/2(2X) = (-1)1/2$

$\underline{X = -1/2}$

Thus, the three values for X in the equatio are:
$X = 0$
$X = -2/3$
$X = -1/2$

STEP E
(Optional)
Check
solution

$24(0) + 28(0) + 8(0) = 0$

$0 + 0 + 0 = 0$

To check if these are the correct values, insert each of the three X values into the original equation and check to see if your answer is 0.

$24(-2/3)^3 + 28(-2/3)^2 + 8(-2/3) = 0$

$24(-8/27) + 28(4/9) + 8(-2/3) = 0$

$24\ (-8/27) + 28\ (12/27) + 8(-18/27) = 0$

$24(-1/2)^3 + 28(-1/2)^2 + 8(-1/2) = 0$

$24(-1/8) + 28(1/4) + -8/2 = 0$

Now you try to solve for all X values in the equation:

$18X^3 + 42X^2 + 12X = 0$

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STEP A
Determine the number of values for X

$18X^{3} + 42X^2 + 12X = 0$

There are 3 values to solve for, since 3 is the highest exponent.

STEP B
Factor out common elements

$3X(6X^2+14X+4) = 0$

Since the variable X is common to all the terms of the equation, it can be factored out. Also, a highest common number to all the terms, 3, can be factored out.

Step C
Simplify the equation

$3X(\quad)(\quad) = 0$

Set up your parentheses.

$3X(\overset{1}{2X} + \overset{2}{4})(\overset{3}{3X} + \overset{4}{1}) = 0$

Put in locations 1 and 3 factors of the 1st term of the equation within the parentheses in step B ($6X^2$), and in location 2 and 4 factors of the 3rd term of the equation within the parentheses in Step B (4). Finally, put some signs between these locations. Remember, you need to get a combination of factors and signs that cross multiply to equal the equation you're solving.

$3X(\overset{1}{2X} + \overset{2}{4})(\overset{3}{3X} + \overset{4}{1}) = 0$

$3X(6X^2 + 2X + 12X + 4) = 0$

Cross multiply to check and see if the two expressions are the ones you want. Cross multiply and add together the four products obtained from multiplying the elements in location 1 and 2, 1 and 4, 2 and 3, and 2 and 4.

$3X(6X^2 + 14X + 4) = 0$

Collect terms and check to see if the result equals the equation in Step B.

If the result does not equal the equation in Step B, try other combinations of factors or signs in the parentheses (return to the beginning of Step C).

STEP D
Solve for X values

$3X = 0$
$\underline{X = 0}$

$2X + 4 = 0$
$(2X + 4) - 4 = 0 - 4$
$2X = -4$
$1/2(2X) = (-4)1/2$
$\underline{X = -2}$

$3X + 1 = 0$
$(3X+1) - 1 = 0 - 1$
$3X = -1$
$1/3(3X) = (-1)1/3$
$\underline{X = -1/3}$

If the result equals the Step B equation, solve for the X values by setting each of the expressions in Step C equal to 0. Then solving for X in each of these equations gives us the 3 values for X that solve the initial equation.

The 3 values for X are:

$X = 0$
$X = -2$
$X = -1/3$

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Now you try to solve for all values of X in the equation:

$$40X^3 + 34X^2 + 6X = 0$$

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Here is the solution:

$40X^3 + 34X^2 + 6X = 0$

$2X(20X^2 + 17X + 3) = 0$

$2X(4X + 1)\ (5X + 3) = 0$

$2X\ (20X^2 + 12X + 5X + 3) = 0$

$2X(20X^2 + 17X + 3) = 0$

$2X = 0$
$X = 0$

$4X + 1 = 0$

$(4X + 1) - 1 = 0 - 1$

$4X = -1$

$1/4(4X) = (-1)1/4$

$X = -1/4$

$5X + 3 = 0$

$(5X + 3) - 3 = 0 - 3$

$5X = -3$

$1/5(5X) = (-3)1/5$

$X = -3/5$

STUDY 2

Pretest for Study 2

$X^2 + 6X + 5 = 0$

$3X^3 + 7X^2 + 2X = 0$

$2X^3 + 9X + 4X = 0$

$3X^2 + 3X - 18 = 0$

$X^2 + 5X + 4 = 0$

$4X^3 - 20X^2 + 24X = 0$

$2X^2 + 8X - 10 = 0$

$3X^3 - 18X^2 + 15X = 0$

STUDY 2

Posttest for Study 2

Please solve for all possible values of X in the following equations:

$5X^3 + 16X^2 + 3X = 0$

$4X^2 + 8X - 12 = 0$

$X^2 + 4X + 3 = 0$

$2X^3 + 7X^2 + 3X = 0$

$2X^3 - 12X^2 + 18X = 0$

$X^2 + 6X + 8 = 0$

$2X^2 + 4X - 16 = 0$

$4X^3 - 24X^2 + 20X = 0$

Study 3

Pretest for Study 3

Please solve for all possible values of X in the following equations:

$X^2 + 6X + 5 = 0$

$20X^3 + 34X^2 + 6X = 0$

$2X^3 + 9X^2 + 4X = 0$

$3X^3 + 7X^2 + 2X = 0$

$18X^3 + 42X^2 + 12X = 0$

$X^2 + 5X + 4 = 0$

STUDY 3

Posttest 1: Following Q_1 Instruction

Please solve for all possible values of X in the following equations:

$12X^3 + 28X^2 + 16X = 0$

$2X^3 + 7X^2 + 3X = 0$

$X^2 + 4X + 3 = 0$

$18X^3 + 39X^2 + 15X = 0$

$5X^3 + 16X^2 + 3X = 0$

$X^2 + 6X + 8 = 0$

STUDY 3

Posttest 2: Following C_2 Instructions

Please solve for all possible values of X in the following equations:

$4X^3 + 13X^2 + 3X = 0$

$X^2 + 3X + 2 = 0$

$X^2 + 5X + 6 = 0$

$30X^3 + 33X^2 + 9X = 0$

$36X^3 + 36X^2 + 8X = 0$

$3X^3 + 11X^2 + 6X = 0$

STUDY 3

Posttest 3: Following C_3 Instructions

Please solve for all possible values of X in the following equations:

$20X^3 + 60X^2 + 45X = 0$

$X^2 + 7X + 10 = 0$

$X^2 + 9X + 20 = 0$

$4X^3 + 7X^2 + 3X = 0$

$5X^3 + 12X^2 + 4X = 0$

$24X^3 + 26X^2 + 6X = 0$

STUDY 3

Retention Test

Please solve for all possible values of X in the following equations:

$X^2 + 7X + 12 = 0$

$3X^3 + 8X^2 + 4X = 0$

$16X^3 + 48X^2 + 20X = 0$

$X^2 + 8X + 16 = 0$

$27X^3 + 24X^2 + 12X = 0$

$3X^3 + 5X^2 + 2X = 0$

APPENDIX C

Appendix C

The appendix describes the latent class techniques and modified path analysis procedures used in the present investigation. Latent class models are designed to represent hypotheses about unobserved (latent) variables. The latent class approach can be used to generate maximum likelihood estimates of expected cell ferquencies under the assumption that the model being examined is true. This estimate of any particular response pattern is obtained by computing the joint probability of the response pattern and the latent class for each latent class. The joint probabilities, which are computed iteratively, are then summed across all latent classes and multiplied by the sample size (Goodman, 1974).

This is illustrated by considering two pairs of identical items, A and A' and B and B'. The general unrestricted latent class model for these items asserts that:

$$\pi_{ijkl} = \sum_{t=1}^{T} \pi^{\bar{A}\bar{A}'\bar{B}\bar{B}'X}_{ijklt}$$

where ijkl is the probability of response pattern ijkl (i = 1, 2; j = 1, 2; k = 1, 2; 1 = 1, 2) and $\pi^{\bar{A}\bar{A}'\bar{B}\bar{B}'X}_{ijklt}$ is the joint probability of response pattern ijkl and latent class t (t = 1 to T). The joint probability may be expressed as:

$$\pi^{X}_{t}\,\pi^{\bar{A}X}_{it}\,\pi^{\bar{A}'X}_{jt}\,\pi^{\bar{B}X}_{kt}\,\pi^{\bar{B}'X}_{lt}$$

where π^{X}_{t} is the probability of latent class $\underline{t}$, $\pi^{\bar{A}X}_{it}$ is the conditional probability that item A will be responded to at level $\underline{i}$, given latent class $\underline{t}$ and $\pi^{\bar{A}'X}_{jt}$, $\pi^{\bar{B}X}_{kt}$, and $\pi^{\bar{B}'X}_{lt}$ are similarly defined.

Various kinds of restrictions can be imposed on the latent class models. For example, the concept of a domain was previously mentioned. This indicates that certain classes of learners ought to perform in similar ways across items. Latent class models can represent this type of assumption through certain restrictions. For instance, for the mastery latent class, one may wish to assume that masters pass all the items, while in the nonmastery latent class one might assume failure of all items. These assumptions would be reflected in the following restrictions:

$$\pi^{\bar{A}X}_{21} = \pi^{\bar{A}'X}_{21} = \pi^{\bar{B}X}_{21} = \pi^{\bar{B}'X}_{21} = 1$$

$$\pi^{\bar{A}X}_{12} = \pi^{\bar{A}'X}_{12} = \pi^{\bar{B}X}_{12} = \pi^{\bar{B}'X}_{12} = 1$$

where $\pi^{\bar{A}X}_{21}$ is the probability of failing item A given memberhip in latent class 1 (nonmastery class), and $\pi^{\bar{A}X}_{12}$ is the probability of passing item A given membership in latent class 2 (mastery class). The other conditional response probabilities are similarly defined.

Latent class models are tested by assessing the correspondence between observed cell frequencies and estimates of expected cell frequencies using the chi-squared statistic. Low values of X^2 indicate models which provide an adequate fit to the data. Clifford Clogg (Note 2) has developed a computer program to carry out the iterative process used to generate maximum likelihood estimates to expected cell frequencies, and which computes the X^2 value to test the fit of a model to a data set. Clogg's program was used in the present investigation.

Goodman's (1973) modified path analysis approach is designed to represent causal relations among a set of categorical variables. Like the latent class approach, Goodman path models can be used to generate maximum likelihood estimates of expected cell frequencies under the assumption that the model being tested is true.

Goodman's models are designed to be analogous to procedures such as regression and the analysis of variance based on the general inear model. The Goodman models may be expressed in either a multiplicative or an additive form. The additive version which makes use of natural logarithms is intended to make the models analogous to analysis of variance and regression procedures.

The general modified path analysis model for three variables can be represented as:

$$\phi^{\bar{C}}_{ij} = \beta^{\bar{C}} + \beta^{\bar{A}C}_{i} + \beta^{\bar{B}C}_{j} + \beta^{\bar{A}BC}_{ij}$$

where $\phi^{\bar{C}}_{ij}$ is the natural logarithm of the odds that variable C will be at level K, level 1 as opposed to level 2, when variables A and B are at levels i and j respectively, $\beta^{\bar{C}}$ is the general mean for variable C expressed in logarithmic form, $\beta^{\bar{A}C}_{i}$ is the main effect of variable A, $\beta^{\bar{B}C}_{j}$ is the main effect of variable B, and $\beta^{\bar{A}BC}_{ij}$ is the AB interaction.

$\beta^{\bar{C}}_{ij}$ is a direct function of expected cell frequencies. Maximum likelihood estimates of expected cell frequencies generated under the model being teste tested are used in computing $\beta^{\bar{C}}_{ij}$.